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# Computational Geometry of Positive Definite Quadratic Forms

## Polyhedral Reduction Theories, Algorithms, and Applications

Achill Schürmann

American Mathematical Society

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Providence, Rhode Island

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For Ida, Jannis and Kristina

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# Preface

The *Geometry of Positive Definite Quadratic Forms* is a rich and old subject which arose in the arithmetic studies of quadratic forms. Through the seminal works of Minkowski and Voronoi a century ago, the geometric viewpoint became predominant. The study of arithmetical and inhomogeneous minima of positive definite quadratic forms turned into a study of lattice sphere packings and coverings. Lattices and, more generally, periodic (point) sets are by now widespread in mathematics and its applications. The important monograph "Sphere packings, Lattices and Groups" [**69**] by Conway and Sloane, with its over 100 pages of references, shows exemplarily the influence on other mathematical disciplines. This becomes particularly apparent for the 24-dimensional Leech lattice and its connections to number theory, group theory, coding theory and mathematical physics. Since the complexity of problems grows with the dimension, it is no surprise that over the past decades more and more computer support was used to study higher dimensional lattices and more general structures. Still, the Geometry of Positive Definite Quadratic Forms is an essential tool, not only in the study of lattice sphere packings and coverings.

One aim of this book is to give a nearly self-contained introduction to this beautiful subject. We present the known material with new proofs, which then admit natural generalizations. These extentions of the known theory were mainly targeted to support the study of extreme periodic sets. However, it turned out that the resulting new theory has other applications as well, as for example, the classification of totally real thin number fields. On the way, always an eye is kept on computability; algorithms are developed that allow computer assisted treatments. Using tools from combinatorial, from linear and from convex optimization, many difficult problems become accessible now. This is, for example, demonstrated in the search for new currently best-known lattice sphere coverings and in the classification of 8-dimensional perfect lattices, which previously was thought to be impossible with the known methods.

Although this book deals with classical topics which have been worked on extensively by numerous authors, it shows exemplarily how computers may help to gain new insights. On the one hand it is shown how computer assisted (sometimes heuristic) exploration helps to discover new exceptional structures. In many cases these would probably not have been found without a computer. On the other hand several computer assisted proofs are given, which deal with extraordinarily large data or involve large enumerations. It is shown how proofs can be obtained from numerical results, by postprocessing of roundoff solutions. All of these aspects of computer mathematics are nowadays supported by a growing functionality of computer algebra systems and by an increasing number of reliable small programs for specific purposes. In some cases one has to combine, to supplement and to improve on existing software tools. If solutions for basic tasks are obtained they should be made accessible to the growing community of computer enthusiastic mathematicians. Underlying many of the presented computational results are in particular two such programs: A program for rigorous determinant maximization (including semidefinite programming) allowing exact certified error bounds, and secondly, a program for polyhedral representation conversion under symmetries.

Computer assisted mathematical explorations and proofs are of increasing importance in many areas of modern mathematics. Even close to the topics of this book there have been amazing developments recently. An example is the proof by Hales [**129**], [**130**] of the famous Kepler conjecture. Several exciting results have been obtained in the context of *linear and semidefinite programming bounds* for spherical codes and point sets in Euclidean spaces. There is the new sphere packing bound by Cohn and Elkies [**60**], and based on it, the proof by Cohn and Kumar [**63**] (see also [**61**]) that the Leech lattice gives the best lattice sphere packing in 24 dimensions. There is the proof of Musin [**185**] showing that the kissing number in four dimensions is 24 (see [**197**] for an excellent survey). Shortly after, Bachoc and Vallentin [**6**], gave more general, new bounds on the size of spherical codes. Their works are followed by similar approaches for other problems, using semidefinite programming. As in some parts of this book, these works involve numerical computations which are then turned into mathematical rigorous proofs. Often numerical quests and subsequent mathematical analysis lead to new mathematical insights. A fascinating example is the study of *universally optimal point configurations*, recently invoked by Cohn and Kumar [**62**] (see also [**9**] and [**264**]). Although all of this is happening literally next door to the topics of this book, I decided to keep it focused as it is. Adequate treatments will hopefully fill other books in the near future. For now I encourage the reader to study the great original works.

*Acknowledgments.* This book grew out of lectures held at an Oberwolfach Seminar on Sphere Packings and at the University of Magdeburg, together with parts of research articles which were previously published, in a similar or partially different form (see [**45**], [**95**], [**225**], [**226**], [**228**], [**229**]). I thank my coauthors David Bremner, Francisco Santos Leal and in particular Mathieu Dutour Sikirić and Frank Vallentin for their many contributions and their shared enthusiasm for the subject. I thank Frank, Mathieu, Henry Cohn, Slava Grishukhin, Jeff Lagarias and Jacques Martinet for their very helpful feedback on prior versions.

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September 2008 Achill Schürmann

# Overview

This overview is intended to guide through the topics and results of this book. Definitions and explanations for used terminology can be found with help of the index. Readers not so familiar with the treated topics may perhaps start by looking into the introductory material in Chapter 1 and Sections 2.1, 3.1 and 4.1. Most of the results in this book are described in the language of positive definite quadratic forms because it is the basis for most computations. In the following summary we nevertheless primarily use "lattice-terminology", since it is more common nowadays. References to the literature can be found in the corresponding context of the book.

**Chapters 1 and 2: An introduction.** The first two chapters are mainly introductory and contain, in contrast to the remaining chapters, only a few new results.

In the first chapter we briefly recall the history of the theory of positive definite quadratic forms and its natural connection to the lattice sphere packing problem. Along the way, we introduce to basic notions and results used in the following chapters. In Sections 1.2, 1.3 and 1.4 we review known results of our main aimed at applications: the sphere packing, the sphere covering and the (simultaneous) sphere packing-covering problem. For all three problems there is a notorious open question: *Do there exist dimensions in which lattices do not give optimal configurations?* This problem is one of our main motivations to extend Voronoi's (lattice) reduction theories in Chapter 3 and Chapter 4.

The second chapter deals with several aspects of Minkowski's reduction theory, which in a sense is a prototype for other polyhedral reduction theories. In Section 2.1, we start with a short discussion about reduction in general, and then introduce Minkowski's theory in Section 2.2. We in particular give a previously unknown, non-redundant description of Minkowski's polyhedral reduction domain up to dimension 7. In Section 2.3 we describe relations to Minkowski's successive minima and state a challenging conjecture concerning an improvement of a classical and central theorem in the Geometry of Numbers. In Section 2.4 we end the second chapter with an application of Minkowski reduction to multidimensional continued fraction expansions, used for simultaneous Diophantine approximations.

**Chapter 3: Voronoi's first reduction theory.** In Section 3.1 we start by introducing to the theory of perfect lattices, respectively to "Voronoi's first reduction theory". Based on so-called *Ryshkov polyhedra*, we give complete proofs for the theory and explain Voronoi's algorithm. We provide some background on computational tools (such as the shortest vector problem and isometry tests for lattices), which were recently used to finish the classification of perfect lattices up to dimension 8.

Our treatment of Voronoi's theory is presented in a way such that it can be generalized naturally from a lattice theory to a theory for *m-periodic point sets.* As lattices can be viewed as linear images of $\mathbb{Z}^d$ in $\mathbb{R}^d$, $m$-periodic sets are linear images of a *standard periodic set*

$$\bigcup_{i=1}^{m} \boldsymbol{t_i} + \mathbb{Z}^d \qquad \text{with} \qquad \boldsymbol{t}_i \in \mathbb{R}^d. \tag{0.1}$$

In Section 3.2, we introduce a new parameter space $\mathcal{S}^{d,m}_{>0}$ to deal with $m$-periodic sets in $\mathbb{R}^d$ up to isometries. We introduce *generalized Ryshkov sets*, on which determinant minimization yields the densest $m$-periodic sphere packings.

In Section 3.3 we analyze local optima ($m$-extreme periodic sets) of the packing density in $\mathcal{S}^{d,m}_{>0}$. We obtain necessary and sufficient conditions. It turns out that our framework provides a new explanation for a previously by Conway and Sloane observed phenomenon; namely for the existence of uncountably many 9-dimensional 2-periodic sphere packings (fluid diamond packings) which are as dense as the densest known lattice sphere packing. We investigate the possibility of improving the densest lattices (represented as points in $\mathcal{S}^{d,m}_{>0}$) locally, to obtain a periodic non-lattice set with larger packing density. We show that this is not possible (see Corollary 3.18). More generally, we show in Theorem 3.17 that a perfect, strongly eutactic lattice is *periodic extreme*, that is, it cannot locally be improved to a denser periodic point set.

In Section 3.4 we give extensions of Voronoi's algorithm, hereby laying the theoretical foundations for systematic, computer assisted searches for dense periodic sphere packings. We develop a "$\boldsymbol{t}$-theory" which enables us to find local sphere packing optima ($\boldsymbol{t}$*-extreme sets*) among all $\boldsymbol{t}$*-periodic point sets.* These are linear images of a standard periodic set (0.1) with a fixed translational part $\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m)$. For the case of a rational matrix $\boldsymbol{t}$ we show in Theorem 3.23, in analogy to the classical theory, that there exist only finitely many $\boldsymbol{t}$-extreme sets. These can be enumerated by a generalized Voronoi algorithm. On top of the $\boldsymbol{t}$-theory, we extend the theory of $G$-perfect and $T$-perfect lattices (where $G$ is a finite subgroup of the orthogonal group and $T$ is a linear subspace in the space of quadratic forms).

This allows us to restrict searches for dense $\boldsymbol{t}$-periodic sphere packings to ones with specific features, for example with a fixed finite symmetry group. For rational $\boldsymbol{t}$ and a finite subgroup of the orthogonal group (leaving (0.1) invariant) we show in Theorem 3.25 that there exist only finitely many local optima among sphere packings with the corresponding properties. This generalizes works of Bergé, Martinet, Sigrist and others. Our proof relies on a general observation (Lemma 3.26) regarding group actions on polyhedral subdivisions. The section ends with three examples of the $G$-theory, describing new results for Eisenstein, Gaussian and Hurwitz quaternionic lattices.

**Chapter 4: Voronoi's second reduction theory.** In the fourth chapter we generalize Voronoi's second reduction theory which is based on Delone polyhedra and Delone subdivisions of lattices. In Section 4.1 we first give the necessary definitions, some background and we explain Voronoi's theory on secondary cones (also called $L$-type domains). We give a simplified and generalized proof for Voronoi's theory in Section 4.2. Theorem 4.7 extends Voronoi's theory to a "$\boldsymbol{t}$-theory" for Delone subdivisions having a standard periodic vertex-set (0.1). For rational $\boldsymbol{t}$ we show that there exist only finitely many non-equivalent secondary cones (see Theorem 4.13). This gives a foundation for systematic searches for thin periodic sphere coverings. Another application is a possible search for "good" periodic quantizers.

In Section 4.3 we introduce an "equivariant $G$-theory" for secondary cones and, more general, a theory of $T$*-secondary cones*, in analogy to the theory of $G$-perfect and $T$-perfect periodic sets described in Section 3.4. As in Section 3.4 we derive from Lemma 3.26 a finiteness result for rational $\boldsymbol{t}$ and finite symmetry groups $G$ (see Theorem 4.19). We put special efforts into an explicit description on how to obtain all Delone subdivisions of lattices and periodic sets within the $G$- and $T$-theory (see in particular Theorem 4.15 and the algorithms in Section 4.3.4). This practicability of our results is of great importance for the applications in Chapter 5.

In Section 4.4 we introduce the secondary cones of single Delone polyhedra and polyhedral complexes. We propose an algorithm to decide whether or not a given simplex with integral vertices is Delone for some positive definite quadratic form. It can possibly serve as a tool to classify lattice Delone simplices (up to $\mathsf{GL}_d(\mathbb{Z})$-equivalence) of a given dimension. We show that the number of inequivalent Delone simplices increases dramatically with the dimension, by constructing Delone simplices with relative volume increasing super-exponentially with the dimension. Previously only linear growth was known.

**Chapter 5: Local analysis of coverings and applications.** In the fifth chapter we harvest the fruits of Chapter 4 and apply them to obtain several results in context of the lattice sphere covering problem. The first two sections have a preparatory character.

In Section 5.1 we formulate algorithms that allow to solve (in principle) the lattice sphere covering and packing-covering problem in a given dimension by solving a finite number of convex optimization problems. We give explicit descriptions of determinant maximization and semidefinite programs. They are subject to linear matrix inequalities expressing that the covering radius of an underlying periodic set is bounded by some given constant (see Proposition 5.5). We describe the software tools we developed to find local lattice covering and packing-covering optima (or at least for generating certified bounds on their covering density or packing-covering constant).

In Section 5.2 we provide tools for a detailed local analysis. These allow to check for local optimality of a lattice, if necessary computationally. Using convex optimization software, we sometimes only have certified ranges for the actual locally optimal value. We show how this restricted information can nevertheless be used to obtain structural informations about the actual local optimizers. We derive quickly computable, local lower bounds for the lattice covering density and the packing-covering constant, applicable to all lattices having a given collection of Delone simplices. For some lattices (as seen for the Leech lattice in Section 5.5) these bounds can be tight (for the right choice of simplices attaining the covering radius). Hence they have the potential to prove local optimality of a lattice.

In Section 5.3 we explain how the tools from the previous two sections can be used to find new best known lattice coverings and packing-coverings. Our techniques enable us in particular to confirm all previously known results up to dimension 5. It is notable that these were previously obtained (without computer assistance) in many years of work and on hundreds of published pages. Beyond the previously known, we extend the knowledge up to dimension 5 by additional information on all local optima. Although the lattice covering and the lattice packing-covering problem have both not been solved in dimension 6 so far, we describe conjecturally optimal lattices obtained by heuristic methods (see Section 5.3.3). Both lattices have the property that their Delone triangulation refines the Delone subdivision of the lattice $\mathsf{E}_6^*$. As a first step in direction of a proof for global optimality, we prove computationally by a branch-and-bound method that the conjectural optimal covering lattice is the unique optimum among lattices having this refinement property (see Theorem 5.28). In many cases it is a problem to obtain exact coordinates for the new lattices. Exemplarily we show how to obtain exact coordinates for the new conjectural optimal 6-dimensional packing-covering lattice. We describe a new conjecturally optimal 7-dimensional covering lattice whose corresponding positive definite quadratic forms are (to our surprise) even rational (see Section 5.3.5). We describe how we obtained new, currently best known covering lattices in dimensions $d \geq 9$. These computations are based on our new $T$-secondary cone theory.

In Section 5.4 we apply the theory of $T$-secondary cones to a problem in algebraic number theory: the classification of all totally real thin algebraic number fields. Our classification is based on a list of 17 candidates, previously given by Bayer-Fluckiger and Nebe. By proving lower bounds on the covering density of positive definite quadratic forms in an associated linear subspaces $T$ we exclude three of the candidates. This finishes the classification.

In Section 5.5 we take a closer look at two of the most exceptional lattices (in low dimensions): The root lattice $\mathsf{E}_8$ and the Leech lattice. We show that both lattices are rigid, meaning they are uniquely determined by the type of their Delone subdivision. By applying our local lower bounds we show that the Leech lattice is a local lattice covering and packing-covering optimum. This shows in particular the existence of rigid, locally optimal lattice coverings, which affirmatively answers a long standing open question of Dickson (1968). The same method can not be applied to the $\mathsf{E}_8$ root lattice. In fact, a local analysis reveals that it is not even a locally optimal covering lattice. In connection with a proof of this fact, we find a new currently best known 8-dimensional covering lattice (see Section 5.5.6).

In Section 5.6 we continue the investigation of lattices similar to $\mathsf{E}_8$, revealing a previously unknown phenomenon: The existence of local lattice covering maxima, for which we derive necessary and sufficient conditions. Based on them we prove computationally that the $\mathsf{E}_6$ root lattice is the only lattice covering maximum in dimensions less or equal to 6, aside of $\mathbb{Z}$ (which trivially is at the same time a 1-dimensional lattice covering optimum). The $\mathsf{E}_8$ root lattice itself turns out to be only “almost” a local lattice covering maximum: Almost any, but not every local change of an associated positive definite quadratic form yields a lower covering density. We baptize these lattices *covering pessima* and prove that a lattice is a covering pessimum, if the Delone polytopes attaining its covering radius are all regular cross polytopes. This is for example the case for the root lattices $\mathsf{D}_4$ and $\mathsf{E}_8$.

In Section 5.7 we consider local lattice covering maxima within a linear subspace $T$ in the space of (associated) quadratic forms. We show a connection to the famous *Minkowski conjecture*, respectively to the so-called *covering conjecture*. Based on our new $T$-secondary cone theory we obtain an algorithm which can be used to verify or falsify the covering conjecture in a given dimension. By a recent result of Curtis McMullen the covering conjecture implies the Minkowski conjecture. Thus (in principle) the Minkowski conjecture can be verified in a given dimension using our algorithm.

**Appendix A: Polyhedral representation conversion.** In Appendix A we provide some background on polyhedra and in particular on a fundamental problem in polyhedral combinatorics, on which many computational results described in this book rely: The representation conversion of polyhedra with large symmetries. We start with a brief review of some basic properties and explain how to compute different polyhedral symmetries. We address the group theoretical background as well as data structures and software tools, which are necessary to deal with large orbits. Finally, we describe Decomposition methods for the representation conversion problem, which have proved to perform best in practice on the polyhedra we treated during our studies.

**Conclusions and prospects.** In this book we describe foundations and some applications of a *Computational Geometry of Positive Definite Quadratic Forms.* Based on generalizations of Voronoi's reduction theories, we provide new algorithms and details on their practical implementation. So far, such implementations have only been used for applications in the context of lattice sphere coverings (see Chapter 5). However, we are convinced that many more, new results can be obtained. Some interesting possible future projects are listed in Appendix B.

We think in particular that (in analogy to the lattice covering problem) many new record breaking lattices for the packing-covering constant and for the so-called *quantizer problem* can be found. We strongly believe that the theory described in this book can be used to find periodic non-lattice sets, which are "better" (with respect to any of the discussed problems) than any lattice of the same dimension. We think that it is only a matter of time and a question of sufficient computational resources, until we will see a solution of the 6-dimensional covering and the 9-dimensional sphere packing problem.

A key ingredient for computational successes described in this book is the representation conversion of polyhedra under symmetry. We think that further improvement on the available tools will not only yield further progress for the problems discussed here, but also for applications beyond the scope of this book. Furthermore, we are convinced that the future will show more and more mathematical proofs based on rigorous numerical computations and non-linear convex optimization. However, there is still plenty of groundwork to be done.

So let's do it...

CHAPTER 1

# From Quadratic Forms to Sphere Packings and Coverings

In this chapter we trace the origin of the lattice sphere packing problem back to the study of Diophantine quadratic equations. We briefly introduce and survey on known results of the sphere packing, the sphere covering and the sphere packing-covering problem. For further reading on these topics and on detailed proofs for the reviewed facts we refer to the books [**206**], [**124**], [**234**], [**69**], [**271**], [**174**], [**121**], and for the history to [**219**] and [**116**].

## 1.1. Positive definite quadratic forms

**1.1.1. Quadratic forms.** In general, a *quadratic form* $Q : V \to K$ is a homogeneous polynomial of degree 2, defined for a field $K$ and a vector space $V$ over $K$. We restrict ourselves in this book to *real or rational quadratic forms* $Q$ defined over $K = \mathbb{R}$ or $K = \mathbb{Q}$. $V$ is throughout a vector space of dimension $d$, hence $V = \mathbb{R}^d$ or $V = \mathbb{Q}^d$. The prefixes "real" or "rational" will not be used, unless necessary. We speak of an *integral quadratic form* $Q$ if its coefficients are integers.

Examples, as the left hand sides of the *Diophantine equations*

$$a^2 + b^2 - c^2 = 0 \quad \text{or} \quad a^2 + b^2 + c^2 + d^2 = n$$

have almost symbolic character and occur throughout the history of mathematics. To find integral solutions for these and other Diophantine equations has fascinated mathematicians from time immemorial, and inspired them to develop beautiful and deep mathematics. It was Fermat, who revived the arithmetic studies of the Greeks and who found many significant results. For example, we attribute him the complete characterization of numbers which are the sum of two squares. The solution to the corresponding three square and four square problem were known to him from the work of Diophantus, but proofs were given much later by Lagrange [**161**] and Legendre [**166**] (cf. [**116**], [**219**]).

Hermite (cf. [**135**]) initiated the systematic arithmetic study of quadratic forms in $d$ variables $\boldsymbol{x} = (x_1, \dots, x_d)^t \in \mathbb{R}^d$. We write

$$Q[\boldsymbol{x}] = \sum_{i,j=1}^{d} q_{ij} x_i x_j$$

with coefficients $q_{ij} \in \mathbb{R}$. By assuming, without loss of generality, that $q_{ij} = q_{ji}$, we simply identify the quadratic form $Q$ with the real symmetric matrix $Q = (q_{ij})_{i,j=1,\dots,d}$. The space of all real quadratic forms in $d$ variables is identified with the space

$$\mathcal{S}^d = \left\{ Q \in \mathbb{R}^{d \times d} : Q^t = Q \right\}$$

of real symmetric $d \times d$ matrices. Using matrix notations we have $Q[\boldsymbol{x}] = \boldsymbol{x}^t Q \boldsymbol{x}$.

For example, the quadratic forms

$$x^2 + y^2 \qquad \text{and} \qquad x^2 + xy + y^2 \tag{1.1}$$

are identified with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}. \tag{1.2}$$

The space $\mathcal{S}^d$ is a $\binom{d+1}{2}$-dimensional real vector space. It becomes a Euclidean space with the usual inner product

$$\langle Q, Q' \rangle = \sum_{i,j=1}^{d} q_{ij} q'_{ij} = \operatorname{trace}(Q \cdot Q'). \tag{1.3}$$

So in particular, we may identify $\mathcal{S}^d$ with $\mathbb{R}^{\binom{d+1}{2}}$.

With the introduced terminology, many **classical problems** involving quadratic forms fit into the following framework: Given a quadratic form $Q \in \mathcal{S}^d$ and $n \in \mathbb{N}$,

$$\boxed{\text{does } Q[\boldsymbol{x}] = n \text{ have integral solutions } \boldsymbol{x} \in \mathbb{Z}^d \text{ ?}} \tag{1.4}$$

**1.1.2. Arithmetical equivalence.** When dealing with this problem in the binary case ($d = 2$), Lagrange noticed that he could study the question up to invertible, linear integral substitutions. Two quadratic forms $Q, Q' \in \mathcal{S}^d$ are called *arithmetically (or integral) equivalent*, if there exists a $U$ in the group

$$\mathsf{GL}_d(\mathbb{Z}) = \{U \in \mathbb{Z}^{d \times d} : |\det U| = 1\}$$

such that

$$Q' = U^t Q U.$$

Arithmetically equivalent forms provide the same set of values on integral vectors, that is, $Q'[\mathbb{Z}^d] = Q[\mathbb{Z}^d]$, because $Q'[\boldsymbol{x}] = Q[U\boldsymbol{x}]$ and $U\mathbb{Z}^d = \mathbb{Z}^d$.

For example, the form

$$3x^2 + 3xy + y^2$$

is arithmetically equivalent to $x^2 + xy + y^2$ because

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}. \tag{1.5}$$

Since for problem (1.4) it is sufficient to deal with one representative of each equivalence class, it is desirable to know a unique form, or at least a small subset of each class. These questions are addressed with in the *theory of reduction* (see Chapter 2).

**1.1.3. Positive definite quadratic forms.** The theory of binary quadratic forms, first systematically studied by Lagrange [**162**], splits into two branches, depending on whether the forms are *indefinite* or *definite*. Because the roots of the lattice sphere packing problem are to be found in the latter case, we restrict ourselves to the definite case from now on.

A quadratic form, respectively real symmetric matrix $Q \in \mathcal{S}^d$, is *positive definite*, if $Q[\boldsymbol{x}] > 0$ for all $\boldsymbol{x} \in \mathbb{R}^d \setminus \{0\}$. The set of all positive definite quadratic forms (PQFs from now on), respectively matrices, is denoted by $\mathcal{S}^d_{>0}$. *Negative definite forms* are just the negative of positive forms and most of what follows just differs by sign.

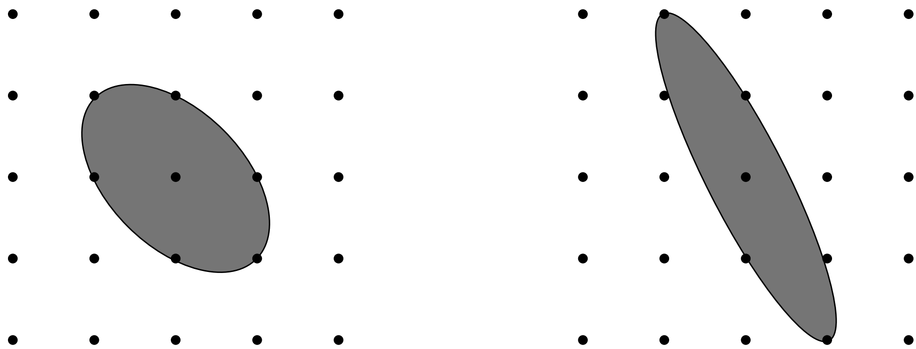

FIGURE 1. Ellipses given by $x^2 + xy + y^2 \leq 1$ and $3x^2 + 3xy + y^2 \leq 1$ .

For PQFs there exists a smallest real number $n$ for which question (1.4) has integral solutions. This number is called the *arithmetical minimum* and denoted by

$$\lambda(Q) = \min_{\boldsymbol{x} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}} Q[\boldsymbol{x}].$$

Sometimes $\lambda(Q)$ is referred to as the *homogeneous minimum.* We have $\lambda(\alpha Q) = \alpha\lambda(Q)$ for a non-negative real scalar $\alpha$.

A PQF $Q$ defines a real valued strictly convex function on $\mathbb{R}^d$ and for $\lambda > 0$

$$E(Q, \lambda) = \{\boldsymbol{x} \in \mathbb{R}^d : Q[\boldsymbol{x}] \leq \lambda\} \tag{1.6}$$

is a non-empty *ellipsoid* with center $\boldsymbol{0}$, providing a geometric interpretation of a PQF. Interpreted geometrically, the arithmetical minimum is the smallest number $\lambda$ for which the ellipsoid $E(Q, \lambda)$ contains an integral point aside of $\boldsymbol{0}$. The integral points $\boldsymbol{x}$ attaining the arithmetical minimum lie on the boundary of the ellipsoid $E(Q, \lambda(Q))$.

$$\operatorname{Min} Q = \{\boldsymbol{x} \in \mathbb{Z}^d : Q[\boldsymbol{x}] = \lambda(Q)\}$$

is the set of *representatives* of the arithmetical minimum.

Arithmetical equivalent PQFs $Q$ and $Q'$, as the two forms in (1.5) for example, have the same arithmetical minimum, and if $Q' = U^t Q U$ for $U \in \mathsf{GL}_d(\mathbb{Z})$, then the sets of representatives satisfy $\operatorname{Min} Q = U \operatorname{Min} Q'$. See Figure 1 for the two ellipses given by the arithmetical equivalent forms in (1.5).

**1.1.4. The space of positive definite quadratic forms.** By the *Sylvester criterion*, $Q \in \mathcal{S}^d$ is positive definite if and only if the upper left minors $Q_k = (q_{ij})_{i,j=1,\ldots,k}$ of $Q$ have positive determinants. Thus

$$\mathcal{S}^d_{>0} = \{Q \in \mathcal{S}^d : \det Q_k > 0, k = 1, \ldots, d\}$$

is given by $d$ polynomial inequalities. It is not hard to see that $\mathcal{S}^d_{>0}$ is an open (full dimensional) *convex cone* in $\mathcal{S}^d$ with apex 0. In particular for $Q \in \mathcal{S}^d_{>0}$, the *open ray* $\{\lambda Q : \lambda > 0\}$ is contained in $\mathcal{S}^d_{>0}$ as well. The cone $\mathcal{S}^d_{>0}$ lies in the *halfspace* $\{Q \in \mathcal{S}^d : \langle Q, \mathrm{id}_d\rangle = \operatorname{trace} Q \geq 0\}$, where $\mathrm{id}_d$ denotes the identity (matrix). The set of PQFs $\mathcal{S}^d_{>0}$ can be thought of as a union of sections $\{Q \in \mathcal{S}^d_{>0} : \operatorname{trace} Q = C\}$ of constant trace, which are dilates of each other. For example, the intersection of $\mathcal{S}^2_{>0}$ (identified with $\mathbb{R}^3$) with a plane of constant trace $\{Q \in \mathcal{S}^2 : \operatorname{trace} Q = C\}$ is a planar disc (ellipse) for $C > 0$ (see Figure 2).

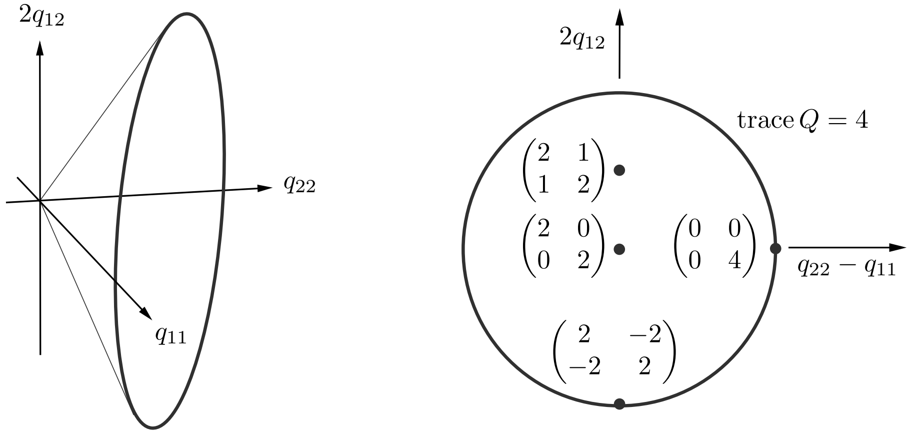


FIGURE 2. Binary PQFs form a convex cone in $\mathbb{R}^3$. PQFs with constant trace lie in an open circle; its boundary contains positive semidefinite forms.

A quadratic form, respectively real symmetric matrix $Q \in \mathcal{S}^d$, is *positive semidefinite* if $Q[\boldsymbol{x}] \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d \setminus \{0\}$. The set of *positive semidefinite forms*, denoted by $\mathcal{S}^d_{\geq 0}$, is the closure of $\mathcal{S}^d_{>0}$. The group $\mathsf{GL}_d(\mathbb{Z})$ acts on $\mathcal{S}^d_{\geq 0}$ and $\mathcal{S}^d_{>0}$ by $Q \mapsto Q[U] = U^t Q U$. Recall from linear algebra that the same group action on $\mathcal{S}^d_{>0}$ by

$$\mathsf{GL}_d(\mathbb{R}) = \left\{ A \in \mathbb{R}^{d \times d} : \det A \neq 0 \right\}$$

has only one orbit, that is, for every $Q \in \mathcal{S}^d_{>0}$ there exists a linear substitution $S \in \mathsf{GL}_d(\mathbb{R})$ with $S^t Q S = \mathrm{id}_d$. We refer to the recent article [**123**] of Gruber for more information on the rich structure of the convex cones $\mathcal{S}^d_{>0}$ and $\mathcal{S}^d_{\geq 0}$.

**1.1.5. Decompositions.** It is well known that every PQF $Q$ can be written as a sum of squares. For example, the *Lagrange expansion* of $Q$ is given by

$$Q[\boldsymbol{x}] = \sum_{i=1}^{d} A_i \left( x_i - \sum_{j=i+1}^{d} \alpha_{ij} x_j \right)^2, \tag{1.7}$$

with unique positive *outer coefficients* $A_i$ and *inner coefficients* $\alpha_{ij} \in \mathbb{R}$, for $i = 1, \ldots, d$ and $j = i+1, \ldots, d$. The inner and outer coefficients provide another parameter space for PQFs. Written as a matrix equation, we obtain the *Jacobi identity* (*Iwasawa decomposition*)

$$Q = S^t D S \tag{1.8}$$

with diagonal matrix $D$ and upper triangular matrix $S$ with diagonal entries equal to 1:

$$D = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_d \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & -\alpha_{12} & \dots & -\alpha_{1d} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\alpha_{d-1,d} \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{1.9}$$

| | | |
|---|---|---|
| $\mathcal{S}^d_{>0}$ | $\overset{\text{bij.}}{\longleftrightarrow}$ | $\mathsf{O}_d(\mathbb{R})\backslash\mathsf{GL}_d(\mathbb{R})$ |
| PQFs | | bases up to orth. trans. |
| $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$ | $\overset{\text{bij.}}{\longleftrightarrow}$ | $\mathsf{O}_d(\mathbb{R})\backslash\overbrace{\mathsf{GL}_d(\mathbb{R})/\mathsf{GL}_d(\mathbb{Z})}^{\text{space of lattices}}$ |
| PQFs up to arith. equiv. | | lattices up to isometries |

FIGURE 3. Connections between lattices and PQFs.

Setting $A = \sqrt{D}S \in \mathsf{GL}_d(\mathbb{R})$ we get a *Cholesky decomposition*

$$Q = A^t A, \tag{1.10}$$

which is uniquely determined by the PQF $Q$ up to orthogonal transformations. That is, if (1.10) holds as well as $Q = B^t B$ for some $B \in \mathsf{GL}_d(\mathbb{R})$, then $A = OB$ with $O$ in

$$\mathsf{O}_d(\mathbb{R}) = \left\{O \in \mathbb{R}^{d\times d} : O^t O = \mathrm{id}_d\right\}.$$

## 1.2. Lattices and the sphere packing problem

**1.2.1. Lattices.** Gauß [**114**] interpreted

$$Q[\boldsymbol{x}] = \boldsymbol{x}^t Q\boldsymbol{x} = \boldsymbol{x}^t A^t A\boldsymbol{x} = \|A\boldsymbol{x}\|^2$$

as the squared (Euclidean) length of the vector $A\boldsymbol{x}$. So the *(point) lattice*

$$L = A\mathbb{Z}^d = \mathbb{Z}\boldsymbol{a}_1 + \cdots + \mathbb{Z}\boldsymbol{a}_d$$

(with column vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d$ of $A$) came into the focus of number theorists. We say $L$ is generated by the *basis* $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$, which we simply identify with the matrix $A \in \mathsf{GL}_d(\mathbb{R})$. Note that lattices are precisely the discrete subgroups of $\mathbb{R}^d$ of rank $d$ (see [**124**]).

The bases of the lattice $L$ are of the form $A' = AU$ with $U \in \mathsf{GL}_d(\mathbb{Z})$. Thus they are in one-to-one correspondence with arithmetical equivalent forms $Q' = (A')^t A' = U^t A^t AU = U^t QU$ of $Q$. Concluding, the revealed connections are shown in Figure 3.

The matrix $A \in \mathsf{GL}_d(\mathbb{R})$ maps the *standard lattice* $\mathbb{Z}^d$ to the lattice $L = A\mathbb{Z}^d$, and the ellipsoid $E(Q, \lambda(Q))$ in (1.6) is transformed to $AE(Q, \lambda(Q)) = \sqrt{\lambda(Q)}B^d$, where

$$B^d = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \le 1\}$$

denotes the *(Euclidean) unit ball* in $\mathbb{R}^d$ with respect to the *Euclidean norm* $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^t\boldsymbol{x}}$. The arithmetical minimum of $Q$ has an interpretation as the squared length of the *shortest non-zero lattice vector*.

The determinant of $Q$ may be geometrically interpreted via the volume $|\det A|$ of the *parallelotope*

$$\{\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_d\boldsymbol{a}_d : \lambda_1, \ldots, \lambda_d \in [0, 1)\}. \tag{1.11}$$

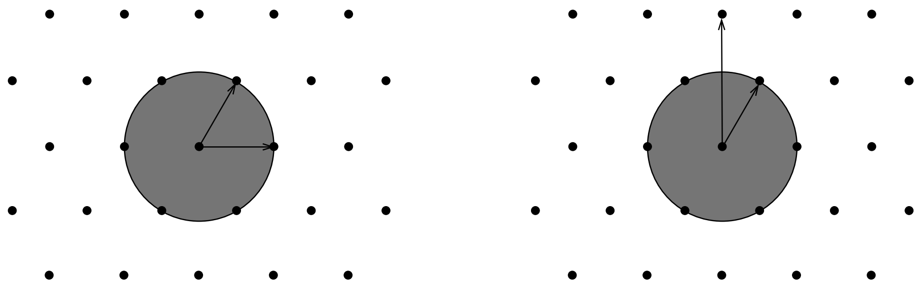

FIGURE 4. Two bases of the *hexagonal lattice* obtained from the PQFs in Figure 1.

The volume is called the *determinant* of the lattice $L = A\mathbb{Z}^d$ and denoted by $\det L$. This is justified, since it does not depend on the choice of the particular basis. For the PQF $Q = A^t A$ we have $\det L = \sqrt{\det Q}$.

**1.2.2. Hermite's constant.** The lattice interpretation of Gauss opened the door for the use of geometric arguments in the theory of quadratic forms. A classical example is the following theorem and its proof.

THEOREM 1.1 (Hermite, [**135**]). *Every lattice $L$ has a base $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d) \in \mathsf{GL}_d(\mathbb{R})$, such that*

$$\|\boldsymbol{a}_1\| \cdots \|\boldsymbol{a}_d\| \leq \left(\frac{4}{3}\right)^{d(d-1)/4} \det L.$$

Using the dictionary, the theorem gives an upper bound for the arithmetical minimum, depending on the determinant of a given PQF $Q$:

COROLLARY 1.2. $\lambda(Q) \leq (\det Q)^{1/d} \cdot \left(\frac{4}{3}\right)^{(d-1)/2}$ *for* $Q \in \mathcal{S}^d_{>0}$.

The corollary in particular implies the existence of *Hermite's constant*

$$\mathcal{H}_d = \sup_{Q \in \mathcal{S}^d_{>0}} \frac{\lambda(Q)}{(\det Q)^{1/d}}. \tag{1.12}$$

Hermite's constant and generalizations have been extensively studied, e.g. in the context of algebraic number theory and differential geometry. We refer to [**32**], [**220**], [**70**] and [**265**] for further reading.

**1.2.3. Lattice sphere packings.** Hermite's theorem 1.1 and its proof (cf. [**174**]) show how helpful the geometric viewpoint can be to obtain arithmetic results as Corollary 1.2. The following sphere packing interpretation due to Minkowski exemplifies this even more.

For a PQF $Q = A^t A$ and a lattice $L = A\mathbb{Z}^d$,

$$\lambda(L) = \frac{\sqrt{\lambda(Q)}}{2}$$

is called the *packing radius* of $L$. Thinking of solid spheres of radius $\lambda(L)$ around each lattice point, the spheres do not overlap and form a *(lattice) sphere packing*

$$L + \lambda(L)B^d = \{\boldsymbol{v} + \lambda(L)\boldsymbol{x} : \boldsymbol{v} \in L, \boldsymbol{x} \in B^d\}. \tag{1.13}$$

See Figure 5 for portions of two famous examples. Due to this interpretation of the arithmetical minimum, the number $|\operatorname{Min} Q|$ of its representatives is also known as

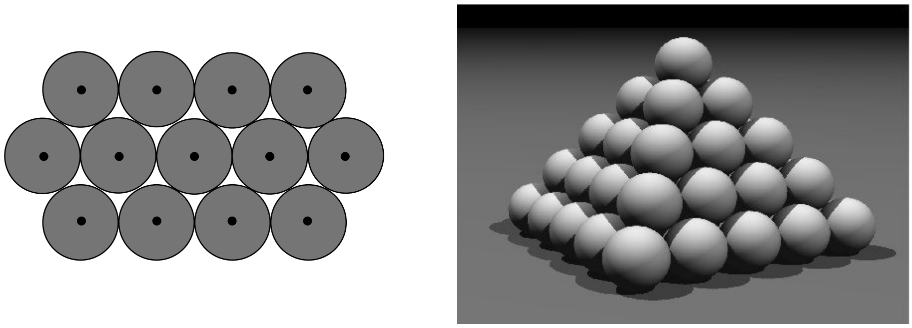

FIGURE 5. Portions of the lattice sphere packings with centers given by the hexagonal and the face-centered-cubic (fcc) lattice.

*kissing number*, because it is the number of spheres in (1.13) which touch ("kiss") a fixed sphere of the packing.

The *(sphere packing) density* $\delta(L)$ of a lattice $L$ is defined as the portion of space covered by spheres, hence

$$\delta(L) = \frac{\operatorname{vol}(\lambda(L)B^d)}{\det L} = \frac{\lambda(L)^d \operatorname{vol} B^d}{\det L}.$$

The PQF $Q$ (and with it any arithmetically equivalent PQF obtained from some basis of $L$) satisfies

$$\delta(L) = \delta(Q) = \left(\frac{\lambda(Q)}{(\det Q)^{1/d}}\right)^{d/2} \cdot \frac{\operatorname{vol} B^d}{2^d}. \tag{1.14}$$

Note that $\delta$ is not only invariant with respect to isometries (of the lattice $L$) and with respect to arithmetical equivalent forms (of the PQF $Q$), but also with respect to scaling. Equation (1.14) shows in particular that the supremum of possible lattice packing densities is, up to a constant factor, equal to a power of Hermite's constant.

Minkowski noticed [**180**] that the trivial bound

$$\delta(Q) \leq 1, \tag{1.15}$$

which is an immediate consequence of the sphere packing interpretation, tremendously improves the upper bound of Hermite for the arithmetical minimum (Corollary 1.2). In fact, (1.15) is equivalent to

$$\lambda(Q) \leq (\det Q)^{1/d} \cdot \frac{4}{(\operatorname{vol} B_d)^{2/d}}. \tag{1.16}$$

showing that the exponential constant on the right in Corollary 1.2 can be replaced by a constant which grows roughly linear with $d$, namely as $\frac{2d}{\pi e}$. This trivial, but significant improvement strengthened the geometric viewpoint and lead Minkowski to a powerful fundamental principle. The ellipsoid $E(Q, r_Q)$, with $r_Q$ being the right hand side in (1.16), has volume

$$\operatorname{vol} E(Q, r_Q) = \operatorname{vol}(\sqrt{r_Q}A^{-1}B^d) = r_Q^{d/2}(\det Q)^{-1/2} \operatorname{vol} B^d = 2^d.$$

Minkowski discovered that not only ellipsoids of volume $2^d$ contain a non-zero integral point, but also all other centrally symmetric *convex bodies* (non-empty, compact convex sets).

| $d$ | lattice | $\delta_d$ | $\mathcal{H}_d$ | author(s) |
|---|---|---|---|---|
| 2 | $\mathsf{A}_2$ | $0.9069\ldots$ | $\left(\frac{4}{3}\right)^{1/2}$ | Lagrange, 1773, [**162**] |
| 3 | $\mathsf{A}_3 = \mathsf{D}_3$ | $0.7404\ldots$ | $2^{1/3}$ | Gauss, 1840, [**114**] |
| 4 | $\mathsf{D}_4$ | $0.6168\ldots$ | $4^{1/4}$ | Korkin & Zolotarev 1877, [**155**] |
| 5 | $\mathsf{D}_5$ | $0.4652\ldots$ | $8^{1/5}$ | Korkin & Zolotarev 1877, [**155**] |
| 6 | $\mathsf{E}_6$ | $0.3729\ldots$ | $\left(\frac{64}{3}\right)^{1/6}$ | Blichfeldt, 1934, [**43**] |
| 7 | $\mathsf{E}_7$ | $0.2953\ldots$ | $64^{1/7}$ | Blichfeldt, 1934, [**43**] |
| 8 | $\mathsf{E}_8$ | $0.2536\ldots$ | 2 | Blichfeldt, 1934, [**43**] |
| 24 | $\Lambda_{24}$ | $0.0019\ldots$ | 4 | Cohn & Kumar, 2004, [**63**] |

TABLE 1. Known optimal lattice packings.

THEOREM 1.3 (Minkowski's Convex Body Theorem). *Any centrally symmetric convex body in $\mathbb{R}^d$ of volume $2^d$ contains a non-zero integral point.*

The theorem is sometimes also called "Minkowski's first (fundamental) theorem". By its background story, the theory of PQFs and lattice sphere packings may be regarded as the root of Minkowski's *Geometry of Numbers* (cf. [**183**]).

**1.2.4. Known results.** The lattice sphere packing interpretation shows that in order to obtain Hermite's constant for a given dimension $d$, we may equivalently try to determine the *density of the densest lattice sphere packing*

$$\delta_d \quad = \sup_{L \subset \mathbb{R}^d \text{ lattice}} \delta(L) \quad = \quad \sup_{Q \in \mathcal{S}^d_{>0}} \delta(Q) \quad = \quad \mathcal{H}_d^{d/2} \frac{\operatorname{vol} B^d}{2^d}.$$

We show in Section 3.1 that this supremum is actually a maximum, which is attained by at most finitely many *densest lattice sphere packings* in a given dimension. By now the lattice sphere packing problem is mainly studied for its own beauty and because of its connections to other fields such as coding and group theory (see [**69**]). Table 1 lists the dimensions in which a solution to the lattice sphere packing problem is known, together with the corresponding densities and values of the Hermite constant.

The lattices $\mathsf{A}_d$ for $d \geq 2$, $\mathsf{D}_d$ for $d \geq 3$ and $\mathsf{E}_d$ for $d = 6, 7, 8$ are the so-called *root lattices.* One of the most fascinating objects is the *Leech Lattice* $\Lambda_{24}$ in 24 dimensions. Definitions and plenty of further information on these fascinating lattices can be found for example in [**69**] (see also [**188**]). Some information on $\mathsf{E}_8$ and $\Lambda_{24}$ is also in Section 5.5. Up to isometries, the lattices in Table 1 give unique optimal solutions to the lattice sphere packing problem. In the 6-dimensional case, this was first shown by Barnes [**25**]. In dimension 7 and 8 uniqueness has been proved by Vetchinkin [**261**]. His paper also contains an extended treatment of Blichfeldt's work (cf. [**266**]). Optimality and uniqueness of the Leech lattice was recently shown by Cohn and Kumar [**63**]. Their methods also provide a second, completely different uniqueness and optimality proof for the lattice $\mathsf{E}_8$. A third proof can be given by application of Voronoi's algorithm. This approach is explained in Section 3.1.

Probably one of the most challenging problems in the theory of (lattice) sphere packings is how the density function behaves asymptotically. It is known to satisfy

$$-1 \lesssim \frac{1}{d}\log_2 \delta_d \lesssim -0.5990\ldots.$$

The lower bound is due to Minkowski (and in a more general form it is known as *Minkowski-Hlawka* bound, cf. the work of Ball [**8**] for the currently "qualitative best known" result and the recent work of Torquato and Stillinger [**253**] for a conjectural improvement). The upper bound is due to Kabatyansky and Levenshtein [**147**], [**170**] (cf. the work of Rogers [**205**] and Cohn and Elkies [**60**], [**59**] for best upper bounds for $d \leq 42$).

**1.2.5. Sphere packings in general.** The lower and upper bounds mentioned above all apply to sphere packings in general, that is, to unions of spheres without the restriction that the centers of spheres form a lattice. Starting with the work of Thue [**251**] (cf. [**108**]) general sphere packings have been studied extensively. A lot of attention, even outside the mathematical community, has been in particular on the 3-dimensional sphere packing problem, also known as *Kepler's conjecture*. After some controversy (see [**141**], [**158**], [**247**], [**107**]; cf. [**223**] for another interpretation of Kepler's original quote), its proof by Hales recently has been published (cf. [**129**] and [**130**]).

In order to define the general sphere packing problem, we need a notion of density. We consider discrete sets $\Lambda$ of $\mathbb{R}^d$, that is, sets which have only finitely many points in any bounded region. Let $C = \{\boldsymbol{x} \in \mathbb{R}^d : |x_i| \leq 1/2\}$ be the $d$-dimensional cube of unit edge length. Then we define the *upper and lower (point) density* of a discrete set $\Lambda$ as

(1.17)

$$\operatorname{dens}_{\sup}\Lambda = \limsup_{\lambda\to\infty}\frac{\operatorname{card}(\Lambda\cap\lambda C)}{\operatorname{vol}\lambda C} \quad\text{and}\quad \operatorname{dens}_{\inf}\Lambda = \liminf_{\lambda\to\infty}\frac{\operatorname{card}(\Lambda\cap\lambda C)}{\operatorname{vol}\lambda C}.$$

If both limits coincide, we simply speak of the point density of $\Lambda$, denoted by $\operatorname{dens}\Lambda$. Note that these notions depend on $C$ and that there exist discrete sets with $\operatorname{dens}_{\sup}\Lambda \neq \operatorname{dens}_{\inf}\Lambda$.

The *packing radius* of a discrete set $\Lambda$ is simply defined by

$$\lambda(\Lambda) = \frac{1}{2}\inf_{\boldsymbol{x},\boldsymbol{y}\in\Lambda,\boldsymbol{x}\neq\boldsymbol{y}}\|\boldsymbol{x}-\boldsymbol{y}\|.$$

The set $\Lambda + \lambda(\Lambda)B^d$ is the *sphere packing defined by* $\Lambda$. In order to have a definition of density available for all discrete sets $\Lambda$, the *sphere packing density* is defined by by

$$\delta(\Lambda) = \lambda(\Lambda)^d \operatorname{vol} B^d \operatorname{dens}_{\inf}\Lambda.$$

The *density of the densest sphere packing* is

$$\delta_d^* = \sup\{\delta(\Lambda) : \Lambda \subset \mathbb{R}^d \text{ a discrete set }\}. \tag{1.18}$$

It turns out that there exist *densest sphere packings*, hence discrete sets $\Lambda$ with $\delta_d^* = \delta(\Lambda)$ (cf. [**137**], [**120**]). For such $\Lambda$, upper and lower point density coincide (cf. also the discussion in [**156**]).

**1.2.6. Periodic sphere packings.** The point density

$$\operatorname{dens}\Lambda = \lim_{\lambda\to\infty} \frac{\operatorname{card}(\Lambda \cap \lambda C)}{\operatorname{vol}\lambda C}$$

exists in particular for *periodic (point) sets*

$$\Lambda = \bigcup_{i=1}^{m} \boldsymbol{t}_i + L,$$

which are given by a lattice $L \subset \mathbb{R}^d$ and $m$ translation vectors $\boldsymbol{t}_i \in \mathbb{R}^d$. In this case we simply have $\operatorname{dens}\Lambda \leq m/\det L$, with equality if the lattice translates $\boldsymbol{t}_i + L$ are pairwise disjoint.

By a **conjecture** of Zassenhaus (cf. [**121**]), $\delta_d^*$ is attained by periodic sets. In Chapter 3 we study periodic sphere packings in greater detail. We in particular consider $\delta$ as a function in a new parameter space, for periodic sets with a fixed number of lattice translates.

One of the biggest **open problems** in the theory of sphere packings is to find dimensions $d$ with $\delta_d < \delta_d^*$, hence dimensions, in which there exist sphere packings which are denser than any lattice packing. There are many possible candidates known, all them periodic packings. For example, among dimensions $d \leq 24$ the densest known sphere packings for $d = 10, 11, 13, 18, 20, 22$ are denser than any known lattice packing (see [**165**], [**259**], [**68**], [**40**]).

## 1.3. The sphere covering problem

The sphere covering problem is, roughly speaking, concerned with the determination of most economical ways to cover $\mathbb{R}^d$ with equally sized spheres. We may think of it as the task to choose a discrete point set in $\mathbb{R}^d$ with a given point density (average number of points in a region of volume 1), such that the maximum distance to points of the surrounding space is as small as possible. The sphere packing problem is sometimes described as the task to distribute a certain number of evil dictators as far apart as possible (*Tammes problem*). In analogy to this, the sphere covering problem may be described as the task to distribute a certain number of friendly governments (or supplies of some sort) such that even the furthest "elements of the surrounding space" are as close as possible to one of them.

Although the problem is as natural as the sphere packing problem it has by far not attracted as many attention. One reason for this are the difficulties arising from the need to include the surrounding space into any consideration. (We learn: For friendly governments the needs of "elements" count.) Another, at first sight astonishing difference, is that large symmetry does not play the same role as it does in the sphere packing problem. For example, the highly symmetric root lattice $\mathsf{E}_8$ is not even a "local optimum" for the lattice covering problem (see Section 5.5).

Nevertheless, the sphere covering problem is fascinating. Answers — many of them yet to be found — give beautiful and exceptional subdivisions of $\mathbb{R}^d$. Table 3 shows some recent progress to construct new currently best known sphere coverings. So it seems, there is still much to discover and underlying principles are yet to be found.

**1.3.1. Density of thinnest sphere coverings.** For a discrete set $\Lambda$ and $\mu \in \mathbb{R}_{>0}$, we speak of a *sphere covering* $\Lambda + \mu B^d = \{\boldsymbol{v} + \mu\boldsymbol{x} : \boldsymbol{v} \in L, \boldsymbol{x} \in B^d\}$ if it

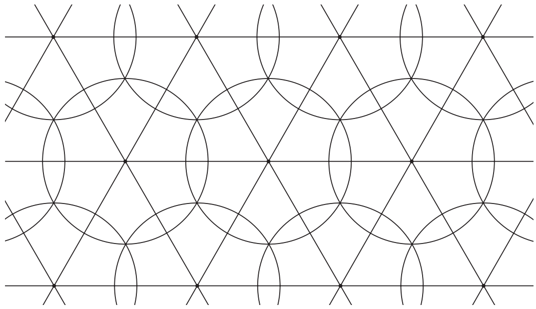

FIGURE 6. Portion of the sphere covering given by the hexagonal lattice.

is equal to all of $\mathbb{R}^d$. The *covering radius* $\mu(\Lambda)$ of $\Lambda$ is defined by

$$\mu(\Lambda) = \inf\{\mu > 0 : \Lambda + \mu B^d = \mathbb{R}^d\}.$$

The *covering density* of $\Lambda$ is defined via the upper point density by

$$\Theta(\Lambda) = \mu(\Lambda)^d \operatorname{vol} B^d \operatorname{dens}_{\sup} \Lambda.$$

Note, in case of a lattice or a periodic set $\Lambda$, we can without problems replace the upper point density by the point density $\operatorname{dens} \Lambda$.

The *density of the thinnest lattice sphere covering* in $\mathbb{R}^d$ is denoted by

$$\Theta_d = \inf\{\Theta(L) : L \subset \mathbb{R}^d \text{ lattice }\},$$

and the *density of the thinnest sphere covering* by

$$\Theta_d^* = \inf\{\Theta(\Lambda) : \Lambda \subset \mathbb{R}^d \text{ a discrete set }\}.$$

As in the case of sphere packings, there exist thinnest lattice coverings and thinnest coverings (cf. [**137**], [**120**]) attaining $\Theta_d$ and $\Theta_d^*$. For such sets, in particular the point density $\operatorname{dens} \Lambda$ is well defined.

As for the sphere packing problem, it has been **conjectured** by Zassenhaus (cf. [**121**]), that $\Theta_d^*$ is attained by periodic sets. As in the case of sphere packings, it is a big **open problem** to determine a dimension $d$ with $\Theta_d > \Theta_d^*$. In other words, it is open whether or not there exists a dimension $d$ with non-lattice coverings thinner than any lattice covering. In particular, if $\Theta_3^* = \Theta_3$ holds is open (the "covering analog of the Kepler conjecture").

**1.3.2. Known results — a snapshot.** The first who considered the sphere covering problem was Kershner, who showed in [**151**] that the hexagonal lattice (see Figure 6) gives the best (thinnest) sphere covering in the planar case, even without the restriction to lattices.

Since then the lattice covering problem has been solved up to dimension 5 (see Table 2). In all these cases the lattice $\mathsf{A}_d^*$ (dual lattice of the root lattice $\mathsf{A}_d$) with

$$\Theta(\mathsf{A}_d^*) = \sqrt{\left(\frac{d(d+2)}{12(d+1)}\right)^d (d+1)} \cdot \operatorname{vol}(B^d),$$

gives the optimal lattice covering. Gameckii [**112**], [**113**], and Bleicher [**42**] were the first to compute the covering density of $\mathsf{A}_d^*$ for general $d$. They also showed that $\mathsf{A}_d^*$ is locally optimal (to be made precise in Section 5.1.1) with respect to lattice covering density in every dimension.

| $d$ | lattice | $\Theta_d$ | author(s) |
|---|---|---|---|
| 2 | $\mathsf{A}_2^*$ | 1.209199 | Kershner [**151**] |
| 3 | $\mathsf{A}_3^*$ | 1.463505 | Bambah [**11**] |
| 4 | $\mathsf{A}_4^*$ | 1.765529 | Delone and Ryshkov [**78**] |
| 5 | $\mathsf{A}_5^*$ | 2.124286 | Ryshkov and Baranovskii |
| | | | [**210**], [**21**], [**22**], [**215**], [**216**] |

TABLE 2. Known optimal lattice coverings.

The lattice optimality of the *body centered cubic lattice* $\mathsf{A}_3^*$ was first proven by Bambah [**11**]. Later, Barnes [**24**] substantially simplified Bambah's proof and strengthened the result by showing that in dimensions 2 and 3 the lattice $\mathsf{A}_d^*$ is the unique locally optimal lattice covering. A third proof was given by Few [**110**]. Nevertheless, it is still an open problem to determine $\Theta_3^*$.

In [**10**] Bambah conjectured that the lattice $\mathsf{A}_4^*$ gives the least dense four-dimensional lattice covering. In [**78**] Delone and Ryshkov proved Bambah's conjecture. In [**13**], [**14**] Baranovskii gave an alternative proof of this fact. He determined all locally optimal lattice coverings in dimension 4. Dickson [**84**] gave another alternative proof of this fact.

In a series of papers [**210**], [**21**], [**22**], [**215**] Ryshkov and Baranovskii solved the lattice covering problem in dimension 5. They prepared a 140-page long monograph [**216**] based on their investigations.

In [**208**] Ryshkov raised the natural question of finding the lowest dimension $d$ for which there is a better lattice covering than the one given by $\mathsf{A}_d^*$. In the same paper he showed that $\mathsf{A}_d^*$ is not the most efficient lattice covering for all even $d \geq 114$ and for all odd $d \geq 201$. Table 3 shows that by now it seems that $\mathsf{A}_d^*$ is likely to give no best lattice coverings for any dimension $d \geq 6$. Most of the results in the table were obtained quite recently, a majority of them based on the theory to be described in Chapter 4. We keep an updated list with additional information on the involved lattices on the web page [**227**].

It may not surprise that the Leech lattice $\Lambda_{24}$ yields the best known lattice covering in dimension 24. Its covering density was computed by Conway, Parker and Sloane (Chapter 23 of [**69**]). Knowing the comparatively low covering density of the Leech lattice, Smith [**239**] was able to estimate the covering densities of the *dual laminated lattices* $\Lambda_{22}^*$ and $\Lambda_{23}^*$. For a definition and more information on laminated lattices and their duals we refer to [**69**]. The lattices $\mathsf{A}_d^r$, with $d \geq 2$ and $r > 1$ divides $d + 1$, are the so-called *Coxeter lattices* (see Section 5.3.6 for a definition). They give local lattice packing optima, but seem not to give locally optimal lattice coverings. Nevertheless, some of them currently give the thinnest known lattice coverings. Information on the lattices $\mathsf{L}_d^c$, some of them obtained by modifying Coxeter lattices, can be found on the web page [**227**].

It is tempting to **conjecture** that the Leech lattice gives a thinnest lattice covering in dimension 24. We show in Section 5.5 that the Leech lattice gives at least a local optimum of the covering function among lattices. This may not be surprising, but on the other hand the root lattice $\mathsf{E}_8$ does not have this property (see Section 5.5). Even worse, $\mathsf{E}_8$ is "almost a local covering maximum". Such "*pessima*" and "real" covering maxima are treated in Section 5.6.

| $d$ | lattice | covering density $\Theta \approx$ | in [**69**] |
|---|---|---|---|
| 6 | $\mathsf{L}_6^c$ | 2.4648 | 2.5511 |
| 7 | $\mathsf{L}_7^c$ | 2.9000 | 3.0596 |
| 8 | $\mathsf{L}_8^c$ | 3.1422 | 3.6658 |
| 9 | $\mathsf{L}_9^c$ | 4.2685 | 4.3889 |
| 10 | $\mathsf{L}_{10}^c$ | 5.1544 | 5.2517 |
| 11 | $\mathsf{L}_{11}^c$ | 5.5182 | 6.2813 |
| 12 | $\mathsf{L}_{12}^c$ | 7.4655 | 7.5101 |
| 13 | $\mathsf{L}_{13}^c$ | 7.7621 | 8.9768 |
| 14 | $\mathsf{L}_{14}^c$ | 8.8252 | 10.727 |
| 15 | $\mathsf{L}_{15}^c$ | 11.0049 | 12.817 |
| 16 | $\mathsf{A}_{16}^*$ | 15.3109 | 15.3109 |
| 17 | $\mathsf{A}_{17}^6$ | 12.3574 | 18.2878 |
| 18 | $\mathsf{A}_{18}^*$ | 21.8409 | 21.8409 |
| 19 | $\mathsf{A}_{19}^{10}$ | 21.2292 | 26.0818 |
| 20 | $\mathsf{A}_{20}^7$ | 20.3668 | 31.1434 |
| 21 | $\mathsf{A}_{21}^{11}$ | 27.7731 | 37.1845 |
| 22 | $\Lambda_{22}^*$ | $\leq 27.8839$ | $\leq 27.8839$ |
| 23 | $\Lambda_{23}^*$ | $\leq 15.3218$ | $\leq 15.3218$ |
| 24 | $\Lambda_{24}$ | 7.903536 | 7.903536 |

TABLE 3. Least dense known (lattice) coverings up to dimension 24.

## 1.4. The sphere packing-covering problem

In the *(simultaneous) packing-covering problem* we ask for discrete sets $\Lambda$ or lattices minimizing the quotient $\Theta(\Lambda)/\delta(\Lambda)$, respectively the *packing-covering constant*

$$\gamma(\Lambda) = \mu(\Lambda)/\lambda(\Lambda).$$

We denote by

$$\gamma_d = \inf\{\gamma(L) : L \subset \mathbb{R}^d \text{ lattice }\}$$

and

$$\gamma_d^* = \inf\{\gamma(\Lambda) : \Lambda \subset \mathbb{R}^d \text{ a discrete set }\}$$

the *smallest lattice packing-covering constant*, respectively the *smallest packing-covering constant.*

The (lattice) packing-covering problem has been studied in different contexts and there are several different names and interpretations of the packing-covering constants $\gamma_d$ and $\gamma_d^*$. Lagarias and Pleasants [**160**] refer to them as the "Delone packing-covering constants". Ryshkov [**211**] studied the equivalent problem of minimizing the density of $(r, R)$-systems. An $(r, R)$-system is a discrete point set $X \subset \mathbb{R}^d$ where (1) the distance between any two points of $X$ is at least $r$ and (2) the distance from any point in $\mathbb{R}^d$ to a point in $X$ is at most $R$. Thus $r = \lambda(X)/2$ and $R = \mu(X)$.

Geometrically we may think of the lattice packing-covering problem as of maximizing the minimum distance between lattice points in a lattice covering with unit spheres. Alternatively, we may think of it as minimizing the radius of a largest sphere that could additionally be packed into a lattice packing of unit spheres (see Figure 7). This minimal gap-radius is equal to $\gamma_d - 1$. Therefore the problem

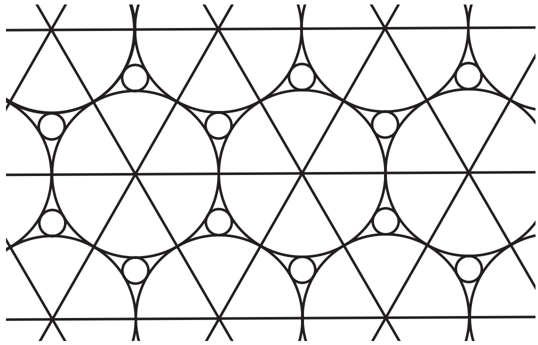

FIGURE 7. A close sphere packing given by the hexagonal lattice.

| $d$ | lattice | $\gamma_d$ | authors |
|---|---|---|---|
| 2 | $\mathsf{A}_2^*$ | $2/\sqrt{3} \approx 1.154700$ | Ryshkov [**211**] |
| 3 | $\mathsf{A}_3^*$ | $\sqrt{5/3} \approx 1.290994$ | Böröczky [**44**] |
| 4 | $\mathsf{Ho}_4$ | $\sqrt{2\sqrt{3}(\sqrt{3}-1)} \approx 1.362500$ | Horváth [**139**] |
| 5 | $\mathsf{Ho}_5$ | $\sqrt{3/2+\sqrt{13}/6} \approx 1.449456$ | Horváth [**140**] |

TABLE 4. Known optimal lattice packing-coverings.

raised by L. Fejes Tóth [**109**] of finding "close packings" attaining this gap-radius is another formulation of the packing-covering problem.

The last interpretation shows that $\gamma_d \geq 2$ would imply that in any $d$-dimensional lattice packing with spheres of unit radius there is still space for spheres of radius 1. Hence, $\delta_d^* \geq 2\delta_d$, which implies of course in particular, that densest sphere packings in dimension $d$ are non-lattice packings. Thus the **open problem** of the existence of $d$ with $\gamma_d \geq 2$ is a very challenging question, in particular in view of the asymptotic bound $\gamma_d \leq 2 + o(1)$ due to Butler [**50**].

As for the lattice covering problem, the lattice packing-covering problem has been solved up to dimension 5 (see Table 4). Ryshkov [**211**] solved the general 2-dimensional case. The 3-dimensional case was settled by Böröczky [**44**], even without the restriction to lattices. The 4- and 5-dimensional lattice cases were solved by Horváth [**139**], [**140**]. Note that the lattices $\mathsf{Ho}_4$ and $\mathsf{Ho}_5$ (see Section 5.3) discovered by Horváth are neither best covering, nor best packing lattices. The results were obtained by using Voronoi's reduction theory, which is described in Section 4.1. So far it is an **open problem** whether or not $\gamma_d^* < \gamma_d$ for any $d \geq 4$.

Our computations, described in Section 5.3, verify all of the known lattice results in dimensions $\leq 5$. As for the covering problem, none of the values $\gamma_d$ has been determined in a dimension $d \geq 6$ so far. In Section 5.3 we report on new best known packing-covering lattices for $d = 6, 7$. We thereby show in particular that $\gamma_6 < 1.412$, revealing the phenomenon $\gamma_6 < \gamma_5$, suspected by Lagarias and Pleasants in [**160**], and observed by Zong [**272**], who showed that $\gamma(\mathsf{E}_6^*) = \sqrt{2} < \gamma_5$.

We were not able yet to find any new best known lattices in dimensions $d \geq 8$. Nevertheless, because of their symmetry and the known bounds on $\gamma_d$, Zong [**272**, Conjecture 3.1] made the following **conjectures**: $\mathsf{E}_8$ and Leech lattice $\Lambda_{24}$ are

| $d$ | lattice | packing covering constant $\gamma \approx$ |
|---|---|---|
| 6 | $L_6^{pc}$ | 1.411081 |
| 7 | $L_7^{pc}$ | 1.494983 |
| 8 | $\mathsf{E}_8$ | 1.414213 |
| 24 | $\Lambda_{24}$ | 1.414213 |

TABLE 5. Lattice packing-covering records.

optimal in their dimensions. In Section 5.5 we show that the Leech lattice gives at least a locally optimal lattice packing-covering constant. Despite some effort, it is an **open question** whether or not the root lattice $\mathsf{E}_8$ gives a locally optimal lattice packing-covering constant as well. The corresponding values of the lattice packing-covering constant are shown in Table 5. The exact value for the smallest known lattice packing-covering constant in dimension 6 is $2\sqrt{2\sqrt{798} - 56}$ (see Section 5.3). In dimensions between 8 and 24 we still do not know enough to state any serious conjectures. We believe that there is still a lot to discover.

CHAPTER 2

# Minkowski Reduction

In this second introductory chapter, we describe Minkowski's reduction theory, which can be used to algorithmically solve the lattice sphere packing problem in a fixed dimension. In Section 2.1 we first consider the task of reduction in general and the binary treatment of Lagrange. Minkowski's reduction, as well as Voronoi's reduction theories, which are the subjects of Chapters 3 and 4, are generalizations of Lagrange's results. In Section 2.2, based on work of Tammela, we obtain a non-redundant description of Minkowski's reduction domain up to dimension 7. Future progress may be obtained by bounding the diagonal entries of reduced positive definite quadratic forms (squared length of lattice basis vectors) in terms of successive minima (squared length of a shortest lattice vectors spanning $\mathbb{R}^d$). Therefore, in Section 2.3, we consider a classical inequality of van der Waerden and conjecture a possible improvement. In Section 2.4 we end the second chapter with a brief description of an application of Minkowski's reduction theory: Lagarias' multidimensional continued fraction expansion used for simultaneous Diophantine approximations.

## 2.1. Reduction in general

In general, the task of reduction is to define a *fundamental domain* (*reduction domain*) $\mathcal{D} \subset \mathcal{S}^d_{>0}$ for the operation $Q \mapsto U^tQU$ of $\mathsf{GL}_d(\mathbb{Z})$ on $\mathcal{S}^d_{>0}$. Such a domain $\mathcal{D}$ satisfies

(i) $\mathcal{S}^d_{>0} = \bigcup\limits_{U \in \mathsf{GL}_d(\mathbb{Z})} U^t\mathcal{D}U,$

(ii) $\operatorname{int}(U^t\mathcal{D}U) \cap \operatorname{int}\mathcal{D} = \emptyset \quad$ for all $\quad U \in \mathsf{GL}_d(\mathbb{Z}) \setminus \{\pm \operatorname{id}\}$.

A PQF $Q \in \mathcal{D}$ is called *reduced.* The purpose of a *reduction algorithm* is to give a procedure which obtains, for any given PQF, an arithmetical equivalent, reduced one. Translated into lattice terminology, we say a lattice basis $A$ of some lattice is *reduced* if $Q = A^tA$ is in $\mathcal{D}$.

Of course, there are many different possibilities to define $\mathcal{D}$. Aside of the polyhedral reduction theories of Minkowski and Voronoi discussed in this book, many other reduction theories have been proposed in the Geometry of Positive Definite Quadratic Forms. Prominent ones are attached to the names of *Hermite* (cf. [**214**]), *Venkov* (cf. [**212**]) and *Korkin-Zolotarev.* Latter has in particular served as a main tool to solve the lattice sphere packing problem (cf. in particular the works of Blichfeldt [**43**] and Vetchinkin [**261**] and the recent computer assisted approach by Pendavingh and van Zwam [**196**]). Note that the popular LLL reduction (cf. [**58**]) is not a reduction in the strict sense used here.

All of the named reduction theories are generalizations of a reduction introduced by Lagrange [**162**] for the binary case. Lagrange showed that

$$\mathcal{D} = \{Q \in \mathcal{S}^2_{>0} : 0 \leq 2q_{12} \leq q_{11} \leq q_{22}\} \tag{2.1}$$

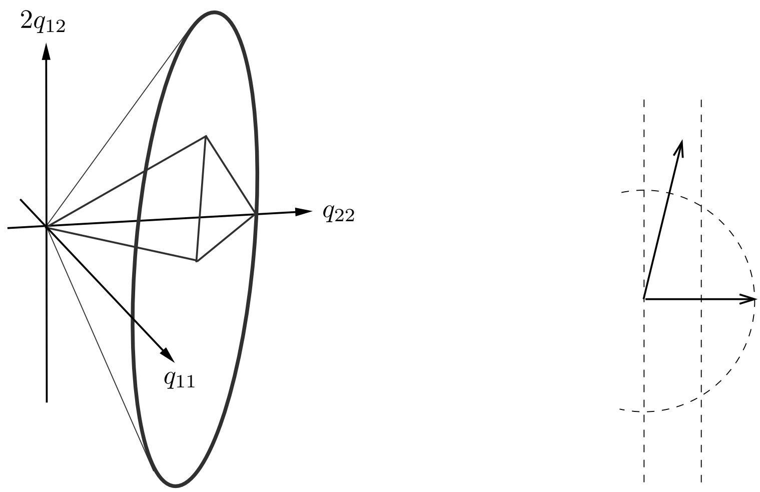


FIGURE 1. Lagrange's reduction domain and a reduced basis.

is a reduction domain in $\mathcal{S}^2_{>0}$. Thinking of lattices in $\mathbb{R}^2$, a base $A = (\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathsf{GL}_2(\mathbb{R})$ is reduced, if $0 \leq \boldsymbol{a}_1^t \boldsymbol{a}_2 \leq \frac{1}{2}\|\boldsymbol{a}_1\|^2$ and $\|\boldsymbol{a}_1\| \leq \|\boldsymbol{a}_2\|$. The first condition implies that the angle between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is between 60 and 90 degrees. By the second condition we see that $\boldsymbol{a}_1$ is the shortest non-zero vector of the lattice.

In Figure 1, $\mathcal{D}$ is shown as a three dimensional polyhedral cone, together with possible positions of a vector $\boldsymbol{a}_2$ in a reduced basis $(\boldsymbol{a}_1, \boldsymbol{a}_2)$. Figure 1 on page 36 shows *Klein's Modulfigur*, which is obtained by intersecting the polyhedral subdivision given by the cones $\{U^t \mathcal{D} U : U \in \mathsf{GL}_d(\mathbb{Z})\}$, with a plane of constant trace.

The reduction constraints can be used to estimate

$$\det Q = q_{11}q_{22} - q_{12}^2 \geq \frac{3}{4} q_{11}$$

for $Q \in \mathcal{D}$ by the arithmetical minimum $\lambda(Q) = q_{11}$. Thus, as an application of Lagrange's reduction theory, we obtain a bound on Hermite's constant

$$\mathcal{H}_2 = \sup_{Q \in \mathcal{S}^2_{>0}} \frac{q_{11}}{(\det Q)^{1/2}} \leq 2/\sqrt{3}.$$

Moreover, the bound is attained only by forms satisfying $2q_{12} = q_{11}$ and $q_{11} = q_{22}$, hence only by positive multiples of $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Consequently, up to similarities, the hexagonal lattice is the unique 2-dimensional lattice maximizing the lattice sphere packing density.

More involved techniques, but in a similar spirit can be used to obtain bounds in other dimensions as well. This applies in particular to Korkine-Zolotarev reduction (cf. [**196**]).

## 2.2. Minkowski reduction

**2.2.1. Minkowski's polyhedral reduction domain.** A generalization of Lagrange's reduction was given by Minkowski.

DEFINITION 2.1. A PQF $Q = (q_{ij}) \in \mathcal{S}^d_{>0}$ is *Minkowski reduced* if

(i) $q_{ii} \leq Q[\boldsymbol{v}]$ for all $\boldsymbol{v} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ such that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{i-1}, \boldsymbol{v}\}$ can be extended to a basis of $\mathbb{Z}^d$, for $i = 1 \dots, d$.

(ii) $q_{i,i+1} \geq 0$ for $i = 1, \dots, d-1$

Note, for a fixed vector $\boldsymbol{v}$, (i) is equivalent to the linear condition

$$\langle Q, \boldsymbol{e}_i \boldsymbol{e}_i^t - \boldsymbol{v}\boldsymbol{v}^t \rangle \leq 0$$

for $Q \in \mathcal{S}^d$. Note also that $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{i-1}, \boldsymbol{v}\}$ can be extended to a basis of $\mathbb{Z}^d$, if and only if $\gcd(\boldsymbol{v}_i, \dots, \boldsymbol{v}_d) = 1$.

A basis $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_d)$ of a lattice is defined to be *Minkowski reduced* if $A^t A$ is Minkowski reduced. This is the case if and only if $\boldsymbol{a}_i$ is a shortest vector among all vectors $\boldsymbol{a}$ such that $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{i-1}, \boldsymbol{a}\}$ can be extended to a basis of $A\mathbb{Z}^d$, and, if in addition $\boldsymbol{a}_i^t \boldsymbol{a}_{i+1} \geq 0$ for $i = 1, \dots, d-1$. In particular, $\boldsymbol{a}_1$ is a shortest non-zero lattice vector. By successively choosing vectors as short as possible while satisfying these conditions, we obtain a Minkowski reduced lattice basis. This shows in particular that every lattice has a Minkowski reduced basis, respectively that every PQF is arithmetically equivalent to a Minkowski reduced one.

The set of Minkowski reduced PQFs form a cone cone in $\mathcal{S}^d_{>0}$ which is defined by the linear inequalities (i) and (ii). Let $\mathcal{M}_d$ denote the convex cone in $\mathcal{S}^d$ defined by the linear inequalities (i) and $\mathcal{M}_d^+$ the one which is defined by the linear inequalities (i) and (ii). The following theorem due to Minkowski shows in particular that among the infinitely many inequalities only finitely many are non-redundant, that is, $\mathcal{M}_d$ and $\mathcal{M}_d^+$ are polyhedral cones (cf. Appendix A).

THEOREM 2.2 (Minkowski [**181**]). *For $d \geq 2$, $\mathcal{M}_d$ and $\mathcal{M}_d^+$ are polyhedral cones in $\mathcal{S}^d_{\geq 0}$. Moreover, $\mathcal{S}^d_{>0} \cap \mathcal{M}_d^+$ is a fundamental domain with respect to the action $Q \mapsto U^t Q U$ of $\mathsf{GL}_d(\mathbb{Z})$ on $\mathcal{S}^d_{>0}$.*

The original proof of Theorem 2.2 by Minkowski in [**181**] has been reconsidered, simplified and generalized by many authors (see for example [**255**], [**234**], [**52**], [**250**]). It is not difficult to see that $\mathcal{M}_d$ is obtained from $\mathcal{M}_d^+$ by an action of the *hyperoctahedral group*:

$$\mathcal{M}_d = \bigcup_{U \in \{\operatorname{diag}(\pm 1, \dots, \pm 1)\}} U^t \mathcal{M}_d^+ U.$$

Because of its symmetry it is in many cases more convenient to work with $\mathcal{M}_d$, instead with $\mathcal{M}_d^+$. For example, although the number of facets of $\mathcal{M}_d$ is larger, we were able to determine it in dimension 7 for $\mathcal{M}_d$, but not for $\mathcal{M}_d^+$ (see Table 2).

**2.2.2. Minkowski's algorithm.** By the following theorem due to Minkowski [**181**] (cf. [**174**] for a proof), Theorem 2.2 has an immediate application to the lattice sphere packing problem.

THEOREM 2.3. *Let $Q, Q' \in \mathcal{S}^d_{>0}$ with $Q \neq Q'$. Then for the set of $\alpha \in \mathbb{R}$ with $Q_\alpha = \alpha Q + (1-\alpha) Q'$ positive definite, $(\det Q_\alpha)^{-1/d}$ is a strictly convex function.*

By Theorem 2.3 the determination of $\delta_d$, respectively $\mathcal{H}_d$, is equivalent to the determination of extreme rays of $\mathcal{M}_d$ or $\mathcal{M}_d^+$. This is due to the fact that the arithmetical minimum $\lambda(Q)$ of PQFs $Q$ is equal to $q_{11}$ in both cones. Hence, $\mathcal{H}_d$ is equal to the minimum of $(\det Q)^{-1/d}$ on the polyhedra

$$\{Q \in \mathcal{M}_d : q_{qq} = 1\} \quad \text{and} \quad \{Q \in \mathcal{M}_d^+ : q_{qq} = 1\}, \tag{2.2}$$

| $x_{i_1}$ | $\pm x_{i_2}$ | $\pm x_{i_3}$ | $\pm x_{i_4}$ | $\pm x_{i_5}$ | $\pm x_{i_6}$ | $\pm x_{i_7}$ | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | |
| 2 | 1 | 1 | 1 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | |
| 2 | 1 | 1 | 1 | 1 | 1 | 0 | |
| 2 | 2 | 1 | 1 | 1 | 1 | 0 | |
| 3 | 2 | 1 | 1 | 1 | 1 | 0 | [1] |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | |
| 2 | 2 | 1 | 1 | 1 | 1 | 1 | |
| 3 | 2 | 1 | 1 | 1 | 1 | 1 | |
| 2 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 3 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 4 | 2 | 2 | 1 | 1 | 1 | 1 | |
| 3 | 3 | 2 | 1 | 1 | 1 | 1 | [2] |
| 4 | 3 | 2 | 1 | 1 | 1 | 1 | [3] |
| 3 | 2 | 2 | 2 | 1 | 1 | 1 | |
| 4 | 3 | 2 | 2 | 1 | 1 | 1 | |

TABLE 1. Tammela's list of linear inequalities defining $\mathcal{M}_d$.

obtained by intersecting $\mathcal{M}_d$ or $\mathcal{M}_d^+$ with the hyperplane $\{Q \in \mathcal{S}^d : q_{11} = 1\}$. By Theorem 2.3, the minimum is attained by vertices of these polyhedra. The vertices are among the uniquely determined forms $Q$ with $\lambda(Q) = q_{11} = 1$ on the extreme rays of $\mathcal{M}_d$ or $\mathcal{M}_d^+$. Note, that some of the rays do not contain positive definite forms and therefore do not give vertices for the polyhedra in (2.2).

This approach to the lattice sphere packing problem is known as *Minkowski's algorithm*. In practice it appears to be more difficult than Voronoi's algorithm, which is based on enumerating perfect PQFs (see Section 3.1). This is due to the fact that every perfect PQF is equivalent to a PQF lying on a ray of $\mathcal{M}_d^+$ (cf. [**209**]). Moreover, by a result of Cohn, Lomakina and Ryshkov [**64**] there exist some rays in $\mathcal{M}_d^+ \subset \mathcal{S}^d_{>0}$ for $d \geq 5$, which contain non-perfect PQFs.

**2.2.3. Non-redundant descriptions.** Minkowski [**179**] gave a list of conditions implying all others in (i) up to dimension 6. Tammela [**249**] enlarged this list to dimension 7. Besides the $d-1$ inequalities $q_{11} \leq \ldots \leq q_{dd}$, the linear conditions for $\mathcal{M}_d$, $d = 2, \ldots, 7$, are attained by plugging the values $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ from Table 1 into (i), where the indices $i_1, \ldots, i_d$ run through all permutations of $\{1, \ldots, d\}$. If $d < 7$ one has to omit the columns $d+1, \ldots, 7$ and the rows with more than $d$ non-zero entries.

We checked these conditions for redundancy using the software `lrs` [**278**] of Avis. Our computations show that in row [1], for $d = 6$, precisely the entries with $x_1 \neq 3$ are redundant (already mentioned by Tammela in [**248**]). For $d = 7$ redundant entries are those with $x_1 \neq 0, 3$ and $x_1 = 0, x_2 \neq 3$. In row [2] the entries

| $d$ | # Facets $\mathcal{M}_d$ | # Rays $\mathcal{M}_d$ | # Facets $\mathcal{M}_d^+$ | # Rays $\mathcal{M}_d^+$ |
|---|---|---|---|---|
| 2 | $3^a$ | $3^a$ | $3^a$ | $3^a$ |
| 3 | $12^a$ | $19^b$ | $9^a$ | $11^b$ |
| 4 | $39^a$ | $323^b$ | $26^b$ | $109^b$ |
| 5 | 200 | 15971 | 117 | 4105 |
| 6 | 1675 | | 1086 | |
| 7 | 65684 | | | |

TABLE 2. Known number of facets and rays of $\mathcal{M}_d$ and $\mathcal{M}_d^+$.

with $x_1 \neq 3$ or $x_2 \neq 3$, and in row $^3$ the entries with $x_1 \neq 4$ or $x_2 \neq 3$ are redundant. The remaining conditions are all non-redundant and define facets of $\mathcal{M}_d$. In Table 2 we list the number of facets and rays as far as we were able to compute them with `cdd` [**275**]. We hereby also confirm earlier results by Minkowski $^a$ [**179**] and Barnes and Cohn $^b$ [**26**].

The computational bottlenecks of Minkowski's approach are apparent. It is not easy to find a sufficiently small system of linear inequalities defining $\mathcal{M}_d$ (or of $\mathcal{M}_d^+$) and even if one has a minimal system of linear inequalities, then computing its rays is a very difficult problem (see Appendix A).

## 2.3. Relations to successive minima

**2.3.1. Successive minima.** In order to get a list of non-redundant inequalities for the description of $\mathcal{M}_d$ and $\mathcal{M}_d^+$, it is desirable to start with a small set of $\boldsymbol{x} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$, possibly defining facets via the conditions $Q[\boldsymbol{x}] \geq Q[\boldsymbol{e}_i]$ (inequality (i) in Definition 2.1). A possible approach, taken by Van der Waerden, is to bound the coefficients of such $\boldsymbol{x}$ in terms of the *successive minima* (cf. [**255**] for details). Here we briefly want to explain how his approximations could possibly be improved. This investigation lead us to a conjecture on the diagonal coefficients (squared length of basis vectors) of a Minkowski reduced form, which we report on.

For $i = 1, \ldots, d$, the $i$-th successive minimum is defined by

$$\lambda_i(Q) = \min\{Q[\boldsymbol{x}_i] : \exists \text{ lin. ind. } \boldsymbol{x}_1, \ldots, \boldsymbol{x}_i \in \mathbb{Z}^d \text{ with } Q[\boldsymbol{x}_1] \leq \ldots \leq Q[\boldsymbol{x}_i]\}. \tag{2.3}$$

Geometrically, we may interpret $\lambda_i$ as the smallest number $\lambda$, such that the ellipsoid $E(Q, \lambda)$ defined in (1.6) contains $i$ linearly independent vectors. For a lattice $L = A\mathbb{Z}^d$ we set

$$\lambda_i(L) = \sqrt{\lambda_i(A^tA)} = \{\|\boldsymbol{v}_i\| : \exists \text{ lin. ind. } \boldsymbol{v}_1, \ldots, \boldsymbol{v}_i \in L \text{ with } \|\boldsymbol{v}_1\| \leq \ldots \leq \|\boldsymbol{v}_i\|\}.$$

In particular, $\lambda_1(Q)$ is the arithmetical minimum of $Q$, respectively $\lambda_1(L)$ is the length of the shortest non-zero vector of $L$. Clearly,

$$\lambda_1(Q) \leq \ldots \leq \lambda_d(Q).$$

By the definition of Hermite's constant, the right hand side in Minkowski's convex body theorem 1.3 for the special case of ellipsoids can (maximally) be sharpened to

$$\lambda_1(Q)^d \leq \mathcal{H}_d^d \det Q.$$

Minkowski observed that the successive minima easily allow further improvement on the left hand side to

$$\lambda_1(Q) \cdots \lambda_d(Q) \leq \mathcal{H}_d^d \det Q \tag{2.4}$$

for $Q \in \mathcal{S}^d_{>0}$ (cf. [**183**]). Note that this is a sharpening of *Minkowski's second fundamental theorem* for the special case of ellipsoids. The corresponding sharpening for arbitrary convex bodies is an open problem, known as *Davenport's conjecture* (see [**121**]).

**2.3.2. A theorem of van der Waerden.** For the proof of Theorem 2.2, Minkowski uses a similar looking inequality

$$q_{11} \cdots q_{dd} \leq \alpha_d \cdot \det Q, \tag{2.5}$$

where $\alpha_d$ is some constant depending on $d$. The exact values for $\alpha_d$ in (2.5) are known for $d \leq 5$ (see [**256**],[**189**]). According to Weyl (cf. [**255**], it is likely that Minkowski unsuccessfully tried to prove (2.5) using (2.4). This missing bridge from Minkowski's Geometry of Numbers to his reduction theory of PQFs was later established by Mahler [**173**] and Weyl [**267**] who gave inequalities

$$q_{ii} \leq \Delta_i \lambda_i(Q)$$

for Minkowski reduced $Q \in \mathcal{S}^d_{>0}$ and for $i = 1, \ldots, d$. Using arguments of Remak [**203**], van der Waerden gave the following, currently best known inequality of this type.

THEOREM 2.4 (van der Waerden, [**255**]). *For a Minkowski reduced PQF $Q \in \mathcal{S}^d_{>0}$ and $i = 1, \ldots, d$ holds*

$$q_{ii} \leq \Delta_i \lambda_i(Q) \quad \textit{with} \quad \Delta_i = \max\left\{1, \left(\frac{5}{4}\right)^{i-4}\right\}.$$

The theorem shows that a $d$-dimensional lattice $L$ has for $d \leq 4$ a (Minkowski reduced) basis $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ with $\|\boldsymbol{a}_i\| = \lambda_i(L)$ for $i = 1, \ldots, d$. This is not the case for $d \geq 5$ as shown by the lattice $L = \mathbb{Z}^d \cup \big((\frac{1}{2}, \ldots, \frac{1}{2})^t + \mathbb{Z}^d\big)$. For its Minkowski reduced basis

$$A = \left(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d-1}, \frac{1}{2}(\boldsymbol{e}_1 + \ldots + \boldsymbol{e}_{d-1})\right) \tag{2.6}$$

and $Q = A^t A$, we obtain $q_{dd} \leq \frac{d}{4}\lambda_d(Q)$. Hence, the inequality of the Theorem is sharp for $d = 5$.

**2.3.3. Improving van der Waerden's theorem?** For $d > 5$ the bound given by Theorem 2.4 seems not best possible. In fact, we **conjecture**

$$q_{ii} \leq \frac{i}{4}\lambda_i(Q) \tag{2.7}$$

for all $i \geq 4$, which would be a tight estimate in case it is true (because of the lattice generated by the Minkowski reduced basis (2.6)).

In principle the conjecture can be verified for a given dimension by checking it for the finitely many rays of $\mathcal{M}_d$ or $\mathcal{M}_d^+$. Although we currently do not know all the rays for dimensions 6 and 7, we made some extensive computational experiments. These seem to indicate that our conjecture is true at least in these dimensions. However, we think with some additional effort, it is possible to give computational proofs for dimensions 6 and 7.

A proof or disproof of (2.7) becomes even more interesting by comparing Minkowski reduced PQFs to Korkin-Zolotarev reduced ones.

THEOREM 2.5 (Lagarias, Lenstra, Schnorr [**159**]). *For a Korkin-Zolotarev reduced $Q \in \mathcal{S}^d_{>0}$ and $i = 1, \ldots, d$ holds*

$$\frac{4}{i+3}\lambda_i(Q) \leq q_{ii} \leq \frac{i+3}{4}\lambda_i(Q).$$

As noted in [**159**], the upper bound is essentially due to Mahler [**172**]. It is linear in $i$, in contrast to the exponential bound of Theorem 2.4 by van der Waerden. We observe the following: Minkowski reduced lattice basis are chosen, taking successively the shortest possible vectors. Korkin-Zolotarev reduced basis are chosen, starting with a shortest non-zero lattice vector $\boldsymbol{a}_1$. Then, for $i = 2, \ldots, d$, successively basis vectors $\boldsymbol{a}_i$ are chosen among lattice vectors $\boldsymbol{a}$ with a shortest non-zero projection $\boldsymbol{a}|L_{i-1}^{\perp}$ (onto the orthogonal complement of the linear subspace $L_{i-1}$ spanned by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}$) such that $\|\boldsymbol{a}_i - (\boldsymbol{a_i}|L_{i-1}^{\perp})\|$ is as small as possible. Loosely spoken, latter condition guarantees that basis vectors are chosen successively "as orthogonal as possible".

Of course, the "greedy strategy" in the construction of Minkowski reduced bases may lead "in the long run" to longer basis vectors than the construction of Korkin-Zolotarev reduced bases does. But that makes a possible disproof for conjecture (2.7) even more interesting.

## 2.4. Lagarias' multidimensional continued fraction expansion

In this section we briefly sketch an application of Minkowski reduction to the classical topic of *Diophantine approximation* through *continued fractions.* Before we describe a multidimensional variant of Lagarias [**157**], which relies on Minkowski's polyhedral domain $\mathcal{M}_d$, we take a look at the classical theory.

**2.4.1. Ordinary continued fractions.** Recall that the *ordinary continued fraction expansion* of a real number $\alpha$ is

$$\alpha = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} \quad ,$$

often with the additional requirement that $a_i \in \mathbb{Z}$ and $a_i \geq 1$ for $i \geq 1$. For details and further reading on ordinary continued fractions we refer to [**133**, Chapter10] and [**153**], or any other introduction to continued fractions.

The *convergents*

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n],$$

with relative prime $p_n$ and $q_n > 0$, are known to give *best (Diophantine) approximations (of the second kind)* in the sense that

$$|p_n - q_n\alpha| < |p - q\alpha| \tag{2.8}$$

for all integers $p$ and $q$ with $1 \leq q \leq q_n$ and $p/q \neq p_n/q_n$; with possible trivial exceptions for $n = 0$ and $\alpha = a_0 + \frac{1}{2}$. Moreover, all such best approximations are convergents to $\alpha$ (see [**153**, Section 6]). These best approximations are also *best approximations of the first kind*, meaning that they minimize $|p/q - \alpha|$ among all integers $p$ and $q$ with $1 \leq q \leq q_n$.

Setting

$$\begin{pmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{pmatrix} = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}$$

it can be easily checked that the *recurrence relation*

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix} \tag{2.9}$$

holds for all $n \geq 0$. By it, all of the matrices (2.9) turn out to have the *unimodularity property*, that is, they are in $\mathsf{GL}_2(\mathbb{Z})$.

We can give a simple geometric interpretation, by considering the integral lattice $\mathbb{Z}^2$, for which the columns of the matrices (2.9) define a basis. The quantity $|p - q\alpha|$ gives the distance of the integral lattice point $(p, q)^t$ to the line $l_\alpha$ through the origin and $(\alpha, 1)^t$. Thus the best approximation property (2.8) shows that $(p_n, q_n)^t$ is closer to (a point of) $l_\alpha$ than any integral point in the strip

$$\left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 : 1 \leq q \leq q_n \right\}. \tag{2.10}$$

Another common interpretation is to look at the family of lattices $L_t = A_t\mathbb{Z}^2$ with basis

$$A_t = \begin{pmatrix} 1 & -\alpha \\ 0 & t \end{pmatrix} \qquad \text{for } t > 0.$$

Relating to the geometric interpretation above: $A_t$ maps $\mathbb{Z}^2$ to $L_t$ and the line $l_\alpha$ to the second coordinate axes. The squared length of the lattice vector $A_t(p, q)^t$ is given by the PQF $Q_t = A_t^t A_t$:

$$Q_t \left[ \begin{pmatrix} p \\ q \end{pmatrix} \right] = (p - q\alpha)^2 + (qt)^2.$$

By the best approximation property (2.8), for small enough $t > 0$ we find

$$Q_t \left[ \begin{pmatrix} p_n \\ q_n \end{pmatrix} \right] < Q_t \left[ \begin{pmatrix} p \\ q \end{pmatrix} \right] \quad \text{for all } \begin{pmatrix} p \\ q \end{pmatrix} \text{ in (2.10).}$$

An idea going back to Hermite [**136**] is to use $\operatorname{Min} Q_t$ for best approximations. In fact, the approximations obtained this way form a (generally strict) subset of best approximation of the second kind, (satisfying 2.8). As $t$ varies, $\operatorname{Min} Q_t$ possibly changes at so-called *Hermite critical values.* On the intervals between these critical values $\operatorname{Min} Q_t$ remains constant; the corresponding integral vectors are called *Hermite approximations.* Hermite showed that consecutive Hermite approximations also have the unimodularity property (give columns of a $\mathsf{GL}_2(\mathbb{Z})$ matrix).

**2.4.2. Multidimensional continued fractions.** *Multidimensional continued fraction (MCF) expansions* are higher dimensional analogies of the ordinary continued fraction expansion preserving the unimodularity property. They assign to some $\boldsymbol{\alpha} \in \mathbb{R}^d$ an initial matrix $U^{(0)} \in \mathsf{GL}_{d+1}(\mathbb{Z})$ and a sequence of *partial quotient matrices* $A^{(i)} \in \mathsf{GL}_{d+1}(\mathbb{Z})$, with $i \geq 0$, giving associated *convergent matrices*

$$U^{(n)} = U^{(0)} A^{(1)} \cdots A^{(n)}.$$

Each column $(p_1, \dots, p_d, q)^t$ of $U^{(n)}$ may be viewed as a *simultaneous Diophantine approximation* to $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)^t$ by rationals $p_1/q, \dots, p_d/q$ with the same denominator $q$. A central problem is to find MCF algorithms that find good Diophantine approximations for any given $\boldsymbol{\alpha} \in \mathbb{R}^d$, where "good" can have several different meanings. We refer to [**47**] and [**230**] for further reading on different MCF expansions and their properties.

Using two consecutive Hermite approximations as columns of matrices $U^{(n)} \in \mathsf{GL}_2(\mathbb{Z})$ gives the so-called *Hermite continued fraction expansion.* One obvious idea to generalize to higher dimensions, is to look at the family of PQFs

$$Q_t = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\alpha_d \\ -\alpha_1 & \cdots & \cdots & -\alpha_d & t^2 + \sum_{i=1}^{d} \alpha_i^2 \end{pmatrix} \in \mathcal{S}_{\geq 0}^{d+1} \quad \text{for } t \geq 0 \tag{2.11}$$

and at corresponding Hermite approximations in $\operatorname{Min} Q_t$. As $t$ varies, $t \to 0$, these give a family of approximations which are a (generally strict) subset of *Euclidean norm best simultaneous approximations*

$$(p_1', \ldots, p_d', q')^t \in \mathbb{Z}^{d+1}.$$

That is, they minimize $\sum_{i=1}^{d} (p_i - q\alpha_i)^2$ over all integral vectors $(p_1, \ldots, p_d, q)^t$ with $1 \leq q \leq q'$. However, in general the unimodularity property of consecutive Hermite approximations is lost; so this does not give an MCF expansion.

An MCF expansion that includes all Hermite approximations at the price of including (many) extra approximations is introduced by Lagarias in [**157**]. For it, consider the open ray $\{Q_t : t > 0\}$ in $\mathcal{S}_{>0}^{d+1}$. It meets (potentially infinitely many) images of Minkowski's polyhedral domain $\mathcal{M}_{d+1}$ in the polyhedral subdivision

$$\left\{ U^t \mathcal{M}_{d+1} U : U \in \mathsf{GL}_{d+1}(\mathbb{Z}) \right\}.$$

For each $t > 0$ there exists a $U_t \in \mathsf{GL}_{d+1}(\mathbb{Z})$ with $U_t^t Q_t U_t \in \mathcal{M}_{d+1}$. As $Q_t$ may be contained in more than one linear image of the cone $\mathcal{M}_{d+1}$, these matrices $U_t$ may not be unique. There is always a finite choice though. It can be constant on segments. By breaking ties in these situations and using the resulting matrices as convergents $U^{(n)}$, we obtain an MCF expansion with only finitely many possible partial quotient matrices $A^{(n)} = (U^{(n-1)})^{-1}(U^{(n)})$. MCF expansions with this property are called *additive.* In contrast to other MCF expansions the possible partial quotients can have negative entries.

The *Lagarias MCF expansion* (called *Minkowski geodesic MCF expansion* by himself) is obtained by resolving the mentioned ambiguities through *lexicographical Minkowski reduced forms.* A Minkowski reduced form is called lexicographical Minkowski reduced, if its diagonal is lexicographically minimal among all arithmetically equivalent Minkowski reduced forms. The notion does not resolve all possible ambiguities, as there exist forms with more than one lexicographical Minkowski reduced form. Nevertheless, Lagarias shows that independent of this remaining ambiguity, there exist uniquely defined *Minkowski critical values* for which the convergents change. These critical values refine the sequence of Hermite critical values. The first column of a convergent always contains a Hermite approximation and all Hermite approximations are obtained in this way. In fact, all columns of the convergents $U^{(n)}$ give rational approximations (by dividing through the last coordinate) that converge "quickly" to any given $\alpha \in \mathbb{R}^d$ for growing $n$. For further details we refer to the excellent article [**157**] of Lagarias and the more detailed exposition in [**134**]. As Lagarias' approach works for other polyhedral reduction theories as well, we think it would be interesting to study the use of Voronoi reduction (see Section 3.1.8) for an MCF expansion.

CHAPTER 3

# Voronoi I

In this chapter, we explain Voronoi's algorithm, which can be used to determine all perfect forms, and among them all local maxima of the sphere packing density function on $\mathcal{S}^d_{>0}$. We in particular report on some background of a computer implementation, which recently allowed us to finish the enumeration of perfect forms in dimension 8. In Section 3.1, we give partially new, simplified proofs for Voronoi's theory based on *Ryshkov polyhedra*. These then allow a very natural generalization of Voronoi's theory to periodic (point) sets in the following sections. In Section 3.2 we consider a new parameter space for periodic sets, given as a union of $m$ lattice translates. In it we investigate in Section 3.3 the behavior of the sphere packing density function and derive sufficient conditions for local optimality. We prove that *strongly eutactic* and *perfect* lattices cannot be locally improved to a denser periodic sphere packing (see Theorem 3.17 and Corollary 3.18). In Section 3.4 we extend the equivariant theory of $G$-perfect forms to periodic sets. In this way we obtain generalizations of Voronoi's classical algorithm, which can be used for systematic computational searches for dense periodic sphere packings. We report on several classification results for *Eisenstein*, *Gaussian* and *Hurwitz quaternionic* perfect forms.

## 3.1. Voronoi's lattice sphere packing legacy

**3.1.1. Ryshkov polyhedra.** In this section we describe an algorithm due to Voronoi, which allows (theoretically) to compute the density $\delta_d$ of the densest lattice sphere packing for a given dimension $d$, by determining all local maxima of $\delta$ on $\mathcal{S}^d_{>0}$. Since $\delta$ is invariant with respect to scaling of PQFs, we may restrict our attention to PQFs with a fixed arithmetical minimum $\lambda > 0$. In order to obtain $\delta_d$, we have to determine the infimum of the determinant $\det Q$ among these PQFs. Or, equivalently, we consider the sets

$$\mathcal{P}_\lambda = \left\{ Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq \lambda \right\} \tag{3.1}$$

and determine Hermite's constant

$$\mathcal{H}_d = \lambda / \inf_{\mathcal{P}_\lambda} (\det Q)^{1/d}.$$

Ryshkov [**209**] noticed that this view on the lattice sphere packing problem allows a simplified treatment of Voronoi's theory to be explained in this section. We follow Gruber [**121**] and refer to $\mathcal{P}_\lambda$ as *Ryshkov polyhedron*. Note that $\mathcal{P}_\lambda = (\lambda/\lambda')\mathcal{P}_{\lambda'}$ for $\lambda, \lambda' > 0$ and that $\mathcal{S}^d_{>0}$ is a union of *arithmetical-minimum-$\lambda$-surfaces*

$$\operatorname{bd} \mathcal{P}_\lambda = \{ Q \in \mathcal{S}^d_{>0} : \lambda(Q) = \lambda \}$$

on which $\mathsf{GL}_d(\mathbb{Z})$ acts by $Q \mapsto U^t Q U$.

Because of

$$Q[\boldsymbol{x}] = \langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle,$$

the quadratic forms $Q \in \mathcal{S}^d$ attaining a fixed value on a given $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ lie all in a *hyperplane* (*affine subspace of co-dimension* 1). The Ryshkov polyhedra $\mathcal{P}_\lambda$ are intersections of infinitely many *halfspaces*:

$$\mathcal{P}_\lambda = \{Q \in \mathcal{S}^d_{>0} : \langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle \geq \lambda \text{ for all } \boldsymbol{x} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}\}. \tag{3.2}$$

In order to describe Voronoi's algorithm geometrically, we need some basic notions from the theory of *polyhedra* (see Appendix A). We say that an intersection of infinitely many halfspaces, $P = \bigcap_{i=1}^\infty H_i^+$, is a *locally finite polyhedron*, if the intersection with an arbitrary polytope is a polytope. So, locally $\mathcal{P}$ "looks like a polytope". One key observation in the theory of lattice sphere packings is the following theorem.

Theorem 3.1. *For $\lambda > 0$ and $d \geq 1$, the Ryshkov polyhedron $\mathcal{P}_\lambda$ (see* (3.2)*) is a locally finite polyhedron.*

Proof. By applying Minkowski's convex body Theorem 1.3, we show that

$$\mathcal{P}_\lambda \cap \{Q \in \mathcal{S}^d : \operatorname{trace} Q \leq C\} \tag{3.3}$$

is a polytope (possibly the empty set) for every constant $C$. This proves the theorem, since

$$\operatorname{trace} Q = \langle Q, \mathrm{id}_d \rangle \leq C$$

determines a halfspace containing a bounded section of $\mathcal{S}^d_{>0}$.

The sets (3.3) are polytopes if the set of all $\boldsymbol{x} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ with $Q[\boldsymbol{x}] = \lambda$ (or $Q[\boldsymbol{x}] \leq \lambda$) for some forms $Q$ in (3.3) is finite. We show below that the absolute value of coordinates

$$m = \max_{i=1,\dots,d} |x_i|$$

of $\boldsymbol{x}$ with this property is bounded.

Let $Q$ be a PQF in (3.3). Then the ellipsoid $E(Q, \lambda) = \{\boldsymbol{x} \in \mathbb{R}^d : Q[\boldsymbol{x}] \leq \lambda\}$ does not contain any point of $\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ in its interior. So in particular $\operatorname{vol} E(Q, \lambda) \leq 2^d$ by Minkowski's convex body theorem. Since

$$\lambda \leq Q[\boldsymbol{e}_i] \leq (\operatorname{trace} Q) - \sum_{j \neq i} Q[\boldsymbol{e}_j] \leq C - (d-1)\lambda,$$

we know that $E(Q, \lambda)$ contains the *cross polytope*

$$C' \cdot \operatorname{conv}\{\pm \boldsymbol{e}_i \ : \ i = 1, \dots, d\} \tag{3.4}$$

(where $\operatorname{conv} M$ denotes the *convex hull* of a set $M$) with

$$C' = \left( \frac{\lambda}{C - (d-1)\lambda} \right)^{1/2}.$$

For $\boldsymbol{x}$ with $Q[\boldsymbol{x}] \leq \lambda$ consider the polytope defined as the convex hull of $\pm\boldsymbol{x}$ and the cross polytope (3.4). It is contained in $E(Q, \lambda)$. On the other hand, this polytope contains the convex hull $P$ of $\pm\boldsymbol{x}$ and the $(d-1)$-dimensional cross polytope

$$C' \cdot \operatorname{conv}\{\pm \boldsymbol{e}_i : i = 1, \dots, d, i \neq j\},$$

where $j \in \{1, \ldots, n\}$ is chosen such that $|x_j|$ attains $m$. Thus setting $C''$ to be the $(d-1)$-dimensional volume of latter $(d-1)$-dimensional cross polytope we get

$$m \cdot \frac{2}{d} C'' = \operatorname{vol} P \leq \operatorname{vol}(\operatorname{conv}\{\pm \boldsymbol{x}, (3.4)\}) < \operatorname{vol} E(Q, \lambda) \leq 2^d.$$

Hence we obtain the desired bound on $m$ (depending only on $\lambda$ and $d$). $\square$

**3.1.2. Forms with constant determinant.** Minimizing the determinant on Ryshkov polyhedra $\mathcal{P}_\lambda$ is simplified by Theorem 2.3 of Minkowski, showing that $(\det Q)^{1/d}$ is a strictly concave function on $\mathcal{S}^d_{>0}$. Note, that in contrast to $(\det Q)^{1/d}$, the function $\det Q$ is not a concave function on $\mathcal{S}^d_{>0}$ (cf. [**189**]). However, the Theorem implies that the set

$$\{Q \in \mathcal{S}^d_{>0} : \det Q \geq D\} \tag{3.5}$$

is strictly convex for $D > 0$. The set of all PQFs $\mathcal{S}^d_{>0}$ is a union of *determinant-$D$-surfaces*

$$\{Q \in \mathcal{S}^d_{>0} : \det Q = D\}$$

on which $\mathsf{GL}_d(\mathbb{Z})$ acts by $Q \mapsto U^t Q U$.

**3.1.3. Perfect forms.** As a consequence of Theorem 2.3 and Theorem 3.1, $\delta_d$ can only be attained at vertices of Ryshkov polyhedra. The vertices $Q$ are called *perfect forms*. They are characterized by the fact that they are uniquely determined by their arithmetical minimum $\lambda = \lambda(Q)$ and its representations $\operatorname{Min} Q$.

EXAMPLE 3.2. $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is perfect, because $\lambda(Q) = 2$ and

$$\operatorname{Min} Q = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

determine $Q$ uniquely through the three linearly independent equations $\langle Q, \boldsymbol{x}\boldsymbol{x}^t \rangle = 2$, $\boldsymbol{x} \in \operatorname{Min} Q$, hence by $q_{11} = 2$, $q_{22} = 2$ and $q_{11} + q_{22} - 2q_{12} = 2$.

In general, $Q$ is perfect if and only if

$$\{\boldsymbol{x}\boldsymbol{x}^t : \boldsymbol{x} \in \operatorname{Min} Q\} \tag{3.6}$$

spans $\mathcal{S}^d$. Thus for perfect $Q$ the kissing number $|\operatorname{Min} Q|$ is at least $(d+1)d$, due to the fact that $\pm\boldsymbol{x}$ in $\operatorname{Min} Q$ give the same *rank*-1 *form* $\boldsymbol{x}\boldsymbol{x}^t$.

The set (3.6) of rank-1 forms generates the *Voronoi domain* (or *Voronoi cone*) of $Q$, denoted by

$$\mathcal{V}(Q) = \operatorname{cone}\{\boldsymbol{x}\boldsymbol{x}^t : \boldsymbol{x} \in \operatorname{Min} Q\}. \tag{3.7}$$

(cone $M$ denotes the conic hull of a set $M$.) The Voronoi domain is full-dimensional if and only if $Q$ is perfect. The Voronoi domain is equal to the *normal cone*

$$\{N \in \mathcal{S}^d : \langle N, Q \rangle \leq \langle N, Q' \rangle \text{ for all } Q' \in \mathcal{P}_{\lambda(Q)}\} \tag{3.8}$$

of $\mathcal{P}_{\lambda(Q)}$ at $Q$.

Since the rank-1 forms in (3.6) cannot span $\mathcal{S}^d$ if the vectors in $\operatorname{Min} Q$ do not span $\mathbb{R}^d$, we obtain the following well known fact, which we use at several places.

PROPOSITION 3.3. *If $Q \in \mathcal{S}^d_{>0}$ is perfect, then* $\operatorname{Min} Q$ *spans* $\mathbb{R}^d$.

In fact, if $\operatorname{Min} Q$ does not span $\mathbb{R}^d$, the set (3.6) can maximally span a $\binom{d}{2}$-dimensional subspace of $\mathcal{S}^d$.

The operation of $\mathsf{GL}_d(\mathbb{Z})$ on $\mathcal{S}^d_{>0}$ leaves $\lambda(Q)$, $\operatorname{Min} Q$ and also $\mathcal{P}_\lambda$ invariant. $\mathsf{GL}_d(\mathbb{Z})$ acts on the sets of faces of a given dimension, thus in particular on the sets of vertices, edges and facets of $\mathcal{P}_\lambda$. A lattice $A\mathbb{Z}^d$ is called *perfect* if $A^tA$ is perfect. Thus bases of perfect lattices correspond to vertices of the locally finite Ryshkov polyhedron $\mathcal{P}_\lambda$ with $\lambda = (2\lambda(L))^2$. The following theorem shows that up to similarities (isometries and scalings), there exist only finitely many perfect lattices in each dimension. That is, the Ryshkov polyhedron $\mathcal{P}_\lambda$ contains only finitely many arithmetically inequivalent vertices. By Theorem 2.3 this implies in particular that $\delta_d$ is actually attained, namely by some perfect lattices, respectively perfect forms.

THEOREM 3.4 (Voronoi 1907). *Up to arithmetical equivalence and scaling there exist only finitely many perfect forms in a given dimension $d \geq 1$.*

PROOF. In the proof of Theorem 3.1 we showed that the set (3.3) of PQFs $Q$ with $\lambda(Q) \geq \lambda > 0$ and $\operatorname{trace} Q \leq C$ is a polytope, hence has only finitely many vertices. Therefore it suffices to show that every perfect PQF $Q$ with $\lambda(Q) = \lambda$ (a vertex of the Ryshkov polyhedron $\mathcal{P}_\lambda$) is arithmetically equivalent to a form with trace smaller than some constant depending only on $\lambda$ and the dimension $d$.

By Hermite's theorem (see Corollary 1.2) we find an equivalent PQF $Q'$ with

$$\prod_{i=1}^{d} q'_{ii} \leq \left(\frac{4}{3}\right)^{d(d-1)/2} \cdot \det Q'. \tag{3.9}$$

The determinant $\det Q' = \det Q$ can be bounded by $\lambda^d$ because of *Hadamard's inequality*, showing

$$\det Q \leq Q[\boldsymbol{a}_1] \cdots Q[\boldsymbol{a}_d] \tag{3.10}$$

for $Q \in \mathcal{S}^d_{>0}$ and linearly independent $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d \in \mathbb{Z}^d$. Latter applies in particular to linearly independent vectors in $\operatorname{Min} Q$, respectively $\operatorname{Min} Q'$ (which exist by Proposition 3.3).

Because of $q'_{ii} \geq \lambda$ we find

$$q'_{kk}\lambda^{d-1} \leq \prod_{i=1}^{d} q'_{ii} \leq \left(\frac{4}{3}\right)^{d(d-1)/2} \lambda^d.$$

From this we obtain the desired upper bound for the trace of $Q'$:

$$\operatorname{trace} Q' = \sum_{k=1}^{d} q'_{kk} \leq d \left(\frac{4}{3}\right)^{d(d-1)/2} \lambda.$$

□

**3.1.4. Voronoi's algorithm.** The vertices (perfect PQFs) and edges of $\mathcal{P}_\lambda$ form the (abstract) *Voronoi graph in dimension d*. Two vertices, respectively perfect PQFs $Q$ and $Q'$ are connected by an edge if the line segment $\operatorname{conv}\{Q, Q'\}$ is an edge of $\mathcal{P}_\lambda$. In this case we say that $Q$ and $Q'$ are *contiguous perfect forms* (or *Voronoi neighbors*). By Theorem 3.4, for given $d$, there are only finitely many vertices (and edges) of the Voronoi graph up to arithmetical equivalence. Therefore, one can enumerate perfect PQFs (up to arithmetical equivalence and scaling) by a *graph traversal algorithm*, which is known as *Voronoi's algorithm* (see Algorithm 1).

**Input:** Dimension $d$.
**Output:** A complete list of inequivalent perfect forms in $\mathcal{S}^d_{>0}$.
Start with a perfect form $Q$.

1. Compute $\operatorname{Min} Q$ and the describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) = \{Q' \in \mathcal{S}^d \ : \ Q'[\boldsymbol{x}] \geq 0 \text{ for all } \boldsymbol{x} \in \operatorname{Min} Q\} \tag{3.11}$$

2. Enumerate extreme rays $R_1, \ldots, R_k$ of the cone $\mathcal{P}(Q)$
3. Determine contiguous perfect forms $Q_i = Q + \alpha R_i$, for $i = 1, \ldots, k$
4. Test if $Q_i$ is arithmetically equivalent to a known form
5. Repeat steps 1.–4. for new perfect forms

ALGORITHM 1. Voronoi's algorithm.

As an initial perfect form we may for example choose *Voronoi's first perfect form*, which is associated to the *root lattice* $\mathsf{A}_d$. For example take $Q_{\mathsf{A}_d} = (q_{i,j})_{1 \leq i,j \leq d}$ with $q_{i,i} = 2$, $q_{i,i-1} = q_{i-1,i} = -1$ and $q_{i,j} = 0$ otherwise (see [**69**, Section 6.1] or [**174**, Section 4.2]).

**3.1.5. Implementing Voronoi's algorithm.** One key ingredient, not only for step 1., is the computation of representations of the arithmetical minimum. For it we may use the *Algorithm of Fincke and Pohst* (cf. [**58**]): Given a PQF $Q$, it allows to compute all $\boldsymbol{x} \in \mathbb{Z}^d$ with $Q[\boldsymbol{x}] \leq C$ for some constant $C > 0$. Note that there exist non-zero vectors $\boldsymbol{x} \in \mathbb{Z}^d$ with $Q[\boldsymbol{x}] \leq \min_{i=1,\ldots,d} q_{ii} =: C$, hence $\lambda(Q) \leq C$. Roughly speaking, the idea is to use the Lagrange expansion (1.7) of $Q$. By it, we may restrict the search to $\boldsymbol{x}$ with

$$\left| x_i - \sum_{i=1}^{d} \alpha_{ij} x_j \right| \leq \sqrt{\frac{C}{A_i}}$$

for $i = d, \ldots, 1$. Here, the bound on the coordinate $x_i$ depends on fixed values of $x_{i+1}, \ldots, x_d$, for which we have only finitely many possible choices. Implementations are provided in computer algebra systems like `MAGMA` [**279**] or `GAP` [**276**] (see also `shvec` by Vallentin [**287**]).

For step 2., observe that the homogeneous cone (3.11) is a translate of the *support cone*

$$\{Q' \in \mathcal{S}^d \ : \ Q'[\boldsymbol{x}] \geq Q[\boldsymbol{x}] \text{ for all } \boldsymbol{x} \in \operatorname{Min} Q\}$$

of $Q$ at $\mathcal{P}_\lambda$. Having its $\mathcal{H}$*-description* (by linear inequalities) we can transform it to its $\mathcal{V}$*-description* and obtain its extreme rays (see Appendix A). The extreme rays $R$ provided by $Q$ through (3.11) are easily seen to be indefinite quadratic forms (see [**174**]).

In step 3., the contiguous perfect forms (Voronoi neighbors) of $Q$ are of the form $Q + \rho R$, where $\rho$ is the smallest positive number such that $\lambda(Q + \rho R) = \lambda$ and $\operatorname{Min}(Q + \rho R) \not\subseteq \operatorname{Min} Q$. It is possible to determine $\rho$, for example with Algorithm 2:

**Input:** A perfect form $Q \in \mathcal{S}^d_{>0}$ and an extreme ray $R$ of (3.11)
**Output:** $\rho > 0$ with $\lambda(Q + \rho R) = \lambda(Q)$ and $\operatorname{Min}(Q + \rho R) \not\subseteq \operatorname{Min} Q$.

```
(l, u) ← (0, 1)
while Q + uR ∉ S^d_{>0} or λ(Q + uR) = λ(Q) do
    if Q + uR ∉ S^d_{>0} then u ← (l + u)/2
    else (l, u) ← (u, 2u)
    end if
end while
while Min(Q + lR) ⊆ Min Q do
    γ ← (l+u)/2
    if λ(Q + γR) ≥ λ(Q) then l ← γ
    else u ← min {(λ(Q) − Q[v])/R[v] : v ∈ Min(Q + γR), R[v] < 0} ∪ {γ}
    end if
end while
ρ ← l
```

Algorithm 2. Determination of Voronoi neighbors.

In phase I (first `while` loop), the procedure determines lower and upper bounds $l$ and $u$ for the desired value $\rho$, such that $Q + lR, Q + uR \in \mathcal{S}^d_{>0}$ with $\lambda(Q + lR) = \lambda$ and $\lambda(Q + uR) < \lambda$. In phase II, the value of $\rho$ is determined. Note that replacing the assignment of $u$ by the simpler assignment $u \leftarrow \gamma$ corresponds to a binary search coming at least arbitrarily close to $\rho$. However, it may never reach the exact value.

For step 4. observe, that based on an algorithm to compute short vectors (for example the one by Fincke-Pohst described above), it is possible to test algorithmically if two PQFs $Q$ and $Q'$ are arithmetically equivalent. That is, because the existence of $U \in \mathsf{GL}_d(\mathbb{Z})$ with $Q' = U^t Q U$ implies

$$q'_{ii} = Q'[\boldsymbol{e}_i] = Q[\boldsymbol{u}_i].$$

Hence for the $i$-th column $\boldsymbol{u}_i$ of $U$ we have only finitely many choices. This idea, but more sophisticated, is implemented in `isom` by Plesken and Souvignier [**198**], which is also part of `MAGMA` [**279**] and `Carat` [**274**]. Note that isometry tests for perfect forms can be simplified, because it suffices to find a $U \in \mathsf{GL}_d(\mathbb{Z})$ with $U \operatorname{Min} Q' = \operatorname{Min} Q$.

Example 3.5. Following the steps above one can show that the 2-dimensional perfect form $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has Voronoi neighbors

$$\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix},$$

which are all arithmetically equivalent to $Q$. Hence, up to scaling and arithmetical equivalence, there exists only one perfect form in $\mathcal{S}^2_{>0}$.

Using the described software tools it is possible to verify the results of Table 1 below on any standard PC up to dimension 6. Note however, that this computation was already done without a computer by Barnes [**25**]. In dimension 7 and beyond

| $d$ | # perf. lat. | # ext. lat. | author(s) |
|---|---|---|---|
| 2 | 1 | 1 | Lagrange, 1773, [**162**] |
| 3 | 1 | 1 | Gauß, 1840, [**114**] |
| 4 | 2 | 2 | Korkin & Zolotarev, 1877, [**155**] |
| 5 | 3 | 3 | Korkin & Zolotarev, 1877, [**155**] |
| 6 | 7 | 6 | Barnes, 1957, [**25**] |
| 7 | 33 | 30 | Jaquet-Chiffelle, 1993, [**143**] |
| 8 | 10916 | 2408 | Dutour Sikirić, Schürmann & Vallentin / Riener, |
| 9 | $> 500000$ | | 2005, [**227**],[**91**],[**204**] |

TABLE 1. Known numbers of perfect and extreme forms.

the explained procedure has a seemingly insuperable "bottleneck": The enumeration of extreme rays for support cones with many facets, respectively for perfect forms with large sets $\operatorname{Min} Q$.

There have been several attempts of using computers to (try to) enumerate perfect forms. Larmouth [**163**] was the first who implemented it and was able to verify the result of Barnes [**25**] up to dimension 6. Also, Stacey [**242**] and Conway and Sloane [**65**] used computer assistance for their attempts to classify the perfect forms in dimension 7. Exploiting symmetries, Jaquet-Chiffelle [**143**] was able to enumerate all perfect forms in dimension 7. Recently, we were able to finish the classification in dimension 8 (see [**227**] and [**91**]).

**3.1.6. An excursion into symmetries.** The recent enumeration successes were previously not possible, because the computation of extreme rays was in particular difficult for the support cones associated to the highly symmetric forms associated to the root lattices $\mathsf{E}_7$ and $\mathsf{E}_8$. Martinet stated that "it seems plainly impossible to classify 8–dimensional perfect lattices" (see [**174**, p.218]). However, it is possible to overcome these difficulties to some extend by exploiting symmetries in the computation of extreme rays (see Appendix A).

In general the *automorphism group* (or *symmetry group*) of a PQF $Q$, or more general of a quadratic form $Q \in \mathcal{S}^d$, is defined by

$$\operatorname{Aut} Q = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U^t Q U = Q\}.$$

As in the case of arithmetical equivalence, we can determine $\operatorname{Aut} Q$, based on the knowledge of all vectors $\boldsymbol{u} \in \mathbb{Z}^d$ with $Q[\boldsymbol{u}] = q_{ii}$ for some $i \in \{1, \dots, d\}$. Again, `MAGMA` [**279**], based on an implementation of Plesken and Souvignier (also available in `Carat` [**274**]), provides a function for this task.

If $L = A\mathbb{Z}^d$ is a lattice and $A^t A = Q$, then every $U \in \operatorname{Aut} Q$ corresponds to an orthogonal transformation $O \in \mathsf{O}_d(\mathbb{R})$ with $AU = OA$. Thus these $U$ define an isometry of the lattice $L$.

EXAMPLE 3.6. For $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ we have

$$\operatorname{Aut} Q = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle$$

with $|\operatorname{Aut} Q| = 12$. The elements of $\operatorname{Aut} Q$ correspond to the 12 linear isometries of the hexagonal lattice.

For $Q \in \mathcal{S}^d_{>0}$, the support cone $\mathcal{P}(Q)$ at $Q$ of the Ryshkov polyhedron $\mathcal{P}_\lambda$ (see (3.11)) and its dual, the Voronoi domain $\mathcal{V}(Q)$, inherit every symmetry of $Q$. That is, for all $U \in \operatorname{Aut} Q$ we have

$$U^t\mathcal{P}(Q)U = \mathcal{P}(Q) \qquad \text{and} \qquad U^t\mathcal{V}(Q)U = \mathcal{V}(Q).$$

EXAMPLE 3.7. In case of the hexagonal lattice with associated PQF $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ we obtain

$$\mathcal{V}(Q) = \operatorname{cone}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\},$$

which is a three dimensional *simplicial cone* with symmetry group $\operatorname{Aut} Q/\{\pm \operatorname{id}_2\}$ of size 6.

The automorphism group of a PQF $Q$ is always finite. On the other hand, for every finite subgroup $G$ of $\mathsf{GL}_d(\mathbb{Z})$, there exists a PQF $Q$ with $G \subseteq \operatorname{Aut} Q$. For example, given an arbitrary $Q' \in \mathcal{S}^d_{>0}$, the PQF

$$Q = \sum_{U \in G} U^t Q' U$$

is invariant with respect to $G$, hence satisfies $G \subseteq \operatorname{Aut} Q$.

Note, for a finite group $G \subset \mathsf{GL}_d(\mathbb{Z})$, the *space of invariant quadratic forms*

$$T_G = \left\{ Q \in \mathcal{S}^d : U^t Q U = Q \text{ for all } U \in G \right\} \tag{3.12}$$

is a linear subspace of $\mathcal{S}^d$; $T_G \cap \mathcal{S}^d_{>0}$ is called *Bravais manifold* of $G$.

EXAMPLE 3.8. In dimension 2, the group $\left\{ \pm \operatorname{id}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ has an associated Bravais manifold with closure cone $\left\{ \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \right\}$.

**3.1.7. Eutaxy and extremality.** Not every perfect form gives a local maximum for the Hermite constant. PQFs giving local maxima for $\delta$ are called *extreme* (cf. [**72**]), and lattices whose bases yield extreme forms are referred to as *extreme lattices*. Note that there is a distinction between extreme and *extremal lattices and forms*, which are only a subclass of the extreme ones (see [**199**] and [**218**] for details). We already saw that extremality (being extreme) implies perfectness, which was first observed by Korkin and Zolotarev in [**155**]. The opposite is not true from dimension 6 onwards (see Table 1).

In order to characterize extreme forms we use the notion of *eutaxy*: A PQF $Q$ is called *eutactic*, if its inverse is contained in the (relative) interior of its Voronoi domain $\mathcal{V}(Q)$:

$$Q^{-1} \in \operatorname{relint} \mathcal{V}(Q).$$

Algebraically, this is equivalent to the existence of positive $\alpha_{\boldsymbol{x}}$ with

$$Q^{-1} = \sum_{\boldsymbol{x} \in \operatorname{Min} Q} \alpha_{\boldsymbol{x}} \boldsymbol{x}\boldsymbol{x}^t. \tag{3.13}$$

Computationally, eutaxy of $Q$ can be tested by solving the *linear program*

$$\max \alpha_{\min} \quad \text{s.t. } \alpha_{\boldsymbol{x}} \geq \alpha_{\min} \text{ and (3.13) holds} \tag{3.14}$$

The form $Q$ is eutactic, if and only if the maximum is greater than 0.

The following theorem of Voronoi shows that perfectness, together with eutaxy implies extremality and vice versa (cf. [**154**]).

THEOREM 3.9 (Voronoi, [**262**]). *A PQF $Q \in \mathcal{S}^d_{>0}$ is extreme if and only if $Q$ is perfect and eutactic.*

PROOF. We give a geometric argument: The function $\det Q$ is a positive real valued polynomial on $\mathcal{S}^d$, depending on the $\binom{d+1}{2}$ coefficients $q_{ij}$ of $Q$. Using the expansion theorem we obtain

$$\det Q = \sum_{i=1}^{d} q^{\#}_{ji} q_{ij}$$

for any fixed column index $j \in \{1, \dots, d\}$. Here, $q^{\#}_{ij} = (-1)^{i+j} \det Q_{ij}$ (with $Q_{ij}$ the minor matrix of $Q$, obtained by removing row $i$ and column $j$) denote the coefficients of the *adjoint form* $Q^{\#} = (\det Q)Q^{-1} \in \mathcal{S}^d_{>0}$ of $Q$. Thus the *gradient* of $\det Q$ satisfies

$$\operatorname{grad} \det Q = (\det Q) Q^{-1}. \tag{3.15}$$

The tangent hyperplane $T$ in $Q$ of the smooth *determinant*-$\det Q$*-surface*

$$S = \{Q' \in \mathcal{S}^d_{>0} : \det Q' = \det Q\}$$

is given by

$$T = \{Q' \in \mathcal{S}^d : \langle Q^{-1}, Q' \rangle = \langle Q^{-1}, Q \rangle\}.$$

Or in other words, $Q^{-1}$ is a normal vector of the tangent plane $T$ of $S$ at $Q$. By Theorem 2.3 we know that $S$ is contained in the halfspace

$$\{Q' \in \mathcal{S}^d : \langle Q^{-1}, Q' - Q \rangle \geq 0\}, \tag{3.16}$$

with $Q$ being the unique intersection point of $S$ and $T$.

As a consequence, a perfect form $Q$ attains a local minimum of $\det Q$ (hence is extreme) if and only if the halfspace (3.16) contains $\mathcal{P}_{\lambda(Q)}$, and its boundary meets $\mathcal{P}_{\lambda(Q)}$ only in $Q$. This is easily seen to be equivalent to the condition that the normal cone (Voronoi domain) $\mathcal{V}(Q)$ of $\mathcal{P}_{\lambda(Q)}$ at $Q$ contains $Q^{-1}$ in its interior. □

For further reading on generalized versions of this classical theorem we refer to [**33**], [**122**] and [**195**]. Note that eutaxy alone does not suffice for extremality. However, there exist only finitely many eutactic forms in every dimension and they can (in principle) be enumerated (see [**174**, Section 9.3]). Nevertheless, this seems computationally more difficult than the enumeration of perfect forms (see [**244**], [**36**], [**31**], [**99**]).

**3.1.8. Voronoi's first reduction.** One possibility to obtain a reduction domain, introduced by Voronoi [**262**], is based on perfect forms and Theorem 3.4. We briefly explain the basic idea.

For $Q \in \mathcal{S}^d_{>0}$ let $\mathcal{V}(Q)$ be the Voronoi domain of $Q$ (see (3.7)). We have $\mathcal{V}(Q) = \mathcal{V}(\lambda Q)$ for $\lambda > 0$ and $\dim \mathcal{V}(Q) = \dim \mathcal{S}^d = \binom{d+1}{2}$ if and only if $Q$ is perfect.

The set of all Voronoi domains forms a poset with respect to inclusion and gives a *polyhedral subdivision* of the *rational closure*

$$\tilde{\mathcal{S}}^d_{\geq 0} = \operatorname{cone}\{\boldsymbol{x}\boldsymbol{x}^t : \boldsymbol{x} \in \mathbb{Z}^d\} \tag{3.17}$$

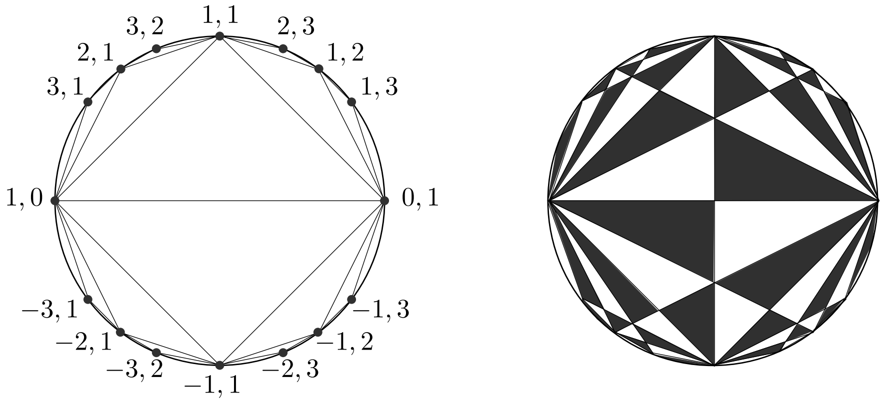


FIGURE 1. Subdivision of $\mathcal{S}^2_{>0}$ by Voronoi domains (left) and "*Klein's Modulfigur*". Numbers $(a, b)$ indicate that the shown semi-definite form is a positive multiple of $(ax + by)^2$.

of $\mathcal{S}^d_{>0}$ (see Proposition 4.1 for another characterization of $\tilde{\mathcal{S}}^d_{\geq 0}$). Here, we understand a polyhedral subdivision to be a decomposition into polyhedra, in which the relative interiors of faces (including the interiors of the polyhedra themselves) do either coincide or do not intersect. This property is referred to as *face-to-face*. By a theorem of Gruber and Ryshkov [**125**], the face-to-face property holds whenever it holds for the facets.

Since only the Voronoi domains of perfect forms are full-dimensional,

$$\tilde{\mathcal{S}}^d_{\geq 0} = \bigcup_{Q \text{ perfect}} \mathcal{V}(Q)$$

and by $\mathcal{S}^d_{>0} \subset \tilde{\mathcal{S}}^d_{\geq 0} \subset \mathcal{S}^d_{\geq 0}$ we obtain a covering of $\mathcal{S}^d_{>0}$ by *perfect (Voronoi) domains* (Voronoi domains of perfect forms). See Figure 1 for a visualization of the 2-dimensional case.

By $Q \mapsto U^t Q U$, the group $\mathsf{GL}_d(\mathbb{Z})$ acts on $\tilde{\mathcal{S}}^d_{\geq 0}$ and on the set of Voronoi domains. By Theorem 3.4 there exist only finitely many perfect domains up to arithmetical equivalence, respectively only finitely many orbits $U^t \mathcal{V} U$. If $\{\mathcal{V}_1, \ldots, \mathcal{V}_k\}$ is a complete system of pairwise inequivalent representatives, then a fundamental domain $\mathcal{D}$ can be constructed as follows: For each $\mathcal{V}_i$ one constructs a fundamental domain $\mathcal{D}_i$ for the action of the finite (automorphism) group $\{U \in \mathsf{GL}_d(\mathbb{Z}) : U^t \mathcal{V}_i U = \mathcal{V}_i\}$ on $\mathcal{V}_i$. Then $\mathcal{D} = \bigcup_{i=1}^k \mathcal{D}_i$ is a fundamental domain of $\mathcal{S}^d_{>0}$ with respect to the action of $\mathsf{GL}_d(\mathbb{Z})$.

EXAMPLE 3.10. For $d = 2$, there exists only one perfect domain up to $\mathsf{GL}_d(\mathbb{Z})$ equivalence. The domain $\mathcal{D}$ by Lagrange in (2.1) is a reduction domain of the Voronoi domain

$$\mathcal{V} = \operatorname{cone}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{of} \quad Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Fundamental domains and hence a possible way of reduction based on perfect forms were explicitly given by Voronoi for $d = 2, 3$ (cf. [**262**]).

Finally, let us remark that perfect forms and Voronoi's (first) reduction theory have several applications to other fields. They are used for *compactifications* of certain *moduli spaces* (cf. for example [**5**], [**232**]), for certain *cohomology*, *homology* and *K-theory* computations (cf. for example [**175**], [**241**], [**99**], [**243**, Appendix]). Moreover, Voronoi's algorithm has been generalized in many different contexts (cf. for example [**37**], [**193**], [**221**]).

## 3.2. Extension to periodic sets

**3.2.1. Parameter spaces for periodic sets.** We want to study the more general situation of periodic sphere packings in greater detail. Recall that a periodic set

$$\Lambda' = \bigcup_{i=1}^{m} \boldsymbol{t}'_i + L \tag{3.18}$$

in $\mathbb{R}^d$ is given by a lattice $L \subset \mathbb{R}^d$, together with $m$ translation vectors $\boldsymbol{t}'_i \in \mathbb{R}^d$, $i = 1, \ldots, m$.

We want to work with a parameter space similar to $\mathcal{S}^d_{>0}$ for lattices. For this, we consider $\Lambda'$ as a linear image $\Lambda' = A\Lambda_{\boldsymbol{t}}$ of a *standard periodic set*

$$\Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d. \tag{3.19}$$

Here, $A \in \mathsf{GL}_d(\mathbb{R})$ satisfies in particular $L = A\mathbb{Z}^d$. Since we are only interested in properties of periodic sets up to isometries, we encode $\Lambda'$ by $Q = A^t A \in \mathcal{S}^d_{>0}$, together with the $m$ translation vectors $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m$. Since every property of periodic sets we deal with here is invariant up to translations, we may assume without loss of generality that $\boldsymbol{t}_m = \boldsymbol{0}$. Thus we consider the *parameter space*

$$\mathcal{S}^{d,m}_{>0} = \mathcal{S}^d_{>0} \times \mathbb{R}^{d\times(m-1)} \tag{3.20}$$

for *m-periodic sets* (up to isometries). We hereby in particular generalize the space $\mathcal{S}^{d,1}_{>0} = \mathcal{S}^d_{>0}$ in a natural way. We call the elements of $\mathcal{S}^{d,m}_{>0}$ *periodic forms* and denote them usually by $X = (Q, \boldsymbol{t})$, where $Q \in \mathcal{S}^d_{>0}$ and

$$\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{m-1}) \in \mathbb{R}^{d\times(m-1)}$$

is a real valued matrix containing $m-1$ columns with vectors $\boldsymbol{t}_i \in \mathbb{R}^d$. One should keep in mind, that although we omit $\boldsymbol{t}_m = \boldsymbol{0}$, we implicitly keep it as a translation vector. Note that a periodic set $\Lambda'$ as in (3.18) has many *representations* by periodic forms. In particular, $m$ may vary and we have different choices for $A$.

The parameter space $\mathcal{S}^{d,m}_{>0}$ is contained in the space

$$\mathcal{S}^{d,m} = \mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}. \tag{3.21}$$

Latter can be turned into a Euclidean space with inner product $\langle\cdot,\cdot\rangle$, defined for $X = (Q, \boldsymbol{t})$ and $X' = (Q', \boldsymbol{t}')$ by

$$\langle X, X'\rangle = \langle Q, Q'\rangle + \sum_{i=1}^{m-1} \boldsymbol{t}_i^t \boldsymbol{t}'_i.$$

Note, for the sake of simplicity we use the same symbol for the inner products on all spaces $\mathcal{S}^{d,m}$.

**3.2.2. The sphere packing problem for periodic sets.** We extend the definition of the arithmetical minimum $\lambda$, by defining the *generalized arithmetical minimum*

$$\lambda(X) = \min\{Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}] : 1 \leq i, j \leq m \text{ and } \boldsymbol{v} \in \mathbb{Z}^d, \text{ with } \boldsymbol{v} \neq \boldsymbol{0} \text{ if } i = j\}$$

for the periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$. Note that we have $\lambda(X) = 0$ in the case of intersecting lattice translates $(\boldsymbol{t}_i + \mathbb{Z}^d) \cap (\boldsymbol{t}_j + \mathbb{Z}^d) \neq \emptyset$ with $i \neq j$. The set of *representations of the generalized arithmetical minimum* $\operatorname{Min} X$ is the set of all $\boldsymbol{w} = \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}$ attaining $\lambda(X)$. Computationally, $\operatorname{Min} X$ and $\lambda(X)$ can be obtained by solving a sequence of *closest vector problems* (CVPs), one for each pair $i, j$ with $i \neq j$. In addition one shortest vector problem (SVP) has to be solved, taking care of the cases where $i = j$. Implementations of algorithms solving CVPs and SVPs are provided for example in `MAGMA` [**279**], `GAP` [**276**] or `shvec` [**287**].

The point density of the periodic set $\Lambda$ represented by a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ is given by

$$\operatorname{dens} \Lambda = m/(\det X)^{\frac{1}{2}},$$

where we set $\det X = \det Q$ for $X = (Q, \boldsymbol{t})$. The (lattice sphere packing) density functions $\delta$ is extended to $\delta : \mathcal{S}^{d,m}_{>0} \to \mathbb{R}$ with

$$\delta(X) = \left(\frac{(\lambda(X))^d}{\det X}\right)^{\frac{1}{2}} m \operatorname{vol} B^d / 2^d.$$

We denote the *density of the densest $m$-periodic sphere packing* by

$$\delta_{d,m} = \sup_{X \in \mathcal{S}^{d,m}_{>0}} \delta(X).$$

Note that $\delta$ is a continuous function on $\mathcal{S}^{d,m}_{>0}$.

We want to take a closer look at local extrema of $\delta_{d,m}$. In analogy to the definition of the Ryshkov polyhedron $\mathcal{P}_\lambda$ in (3.1), we define for $\lambda > 0$ the *(generalized) Ryshkov sets*

$$\mathcal{P}_{m,\lambda} = \left\{X \in \mathcal{S}^{d,m}_{>0} : \lambda(X) \geq \lambda\right\}. \tag{3.22}$$

The condition $\lambda(X) \geq \lambda$ gives infinitely many linear inequalities

$$p_{\boldsymbol{v}}(X) = Q[\boldsymbol{v}] = \langle X, (\boldsymbol{v}\boldsymbol{v}^t, 0)\rangle \geq \lambda$$

for $\boldsymbol{v} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$, as in the case $m = 1$. For $m > 1$ we additionally have the infinitely many polynomial inequalities

$$p_{i,j,\boldsymbol{v}}(X) = Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}] \geq \lambda, \tag{3.23}$$

where $i, j \in \{1, \ldots, m\}$ with $i \neq j$ and $\boldsymbol{v} \in \mathbb{Z}^d$. These polynomials are of degree 3 in the parameters $q_{kl}$, $t_{kl}$ of $X$. Note that they are linear for a fixed $\boldsymbol{t}$. Observe also that $p_{i,m,\boldsymbol{v}}$ and $p_{m,j,\boldsymbol{v}}$ are special due to our assumption $\boldsymbol{t}_m = \boldsymbol{0}$ and that there is a symmetry $p_{i,j,\boldsymbol{v}} = p_{j,i,-\boldsymbol{v}}$ by which we may restrict our attention to polynomials with $i \leq j$. In case of equality $i = j$ we have the linear function $p_{i,i,\boldsymbol{v}} = p_{\boldsymbol{v}}$.

The density of the densest $m$-periodic sphere packing $\delta_{d,m}$ is attained by a periodic form $X$ attaining the "Hermite like constant"

$$\sup_{X \in \mathcal{S}^{d,m}_{>0}} \lambda(X)/(\det X)^{1/d} = \lambda / \inf_{X \in \mathcal{P}_{m,\lambda}} (\det X)^{1/d},$$

where we may choose an arbitrary $\lambda > 0$ on the right hand side. Thus in order to determine $\delta_{d,m}$ we may minimize $(\det X)^{1/d}$ on $\mathcal{P}_{m,\lambda}$.

In analogy to the lattice case, we call a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ *m-extreme*, if it attains a local maximum of $\delta$ on $\mathcal{S}^{d,m}_{>0}$.

## 3.3. Local analysis of periodic sphere packings

**3.3.1. Characterizing local optima.** Before we generalize perfectness and eutaxy to a notion of *m-perfectness* and *m-eutaxy* (in order to obtain a sufficient condition for a periodic form to be $m$-extreme from it) we discuss a rather general setting: Assume we want to optimize a smooth function on a *basic closed semi-algebraic set*, that is, on a region which is described by finitely many (non-strict) polynomial inequalities. Let $E$ denote a Euclidean space with inner product $\langle\cdot,\cdot\rangle$. Further, let $f : E \to \mathbb{R}$ be *smooth* (infinitely differentiable) and $g_1, \dots, g_k$ be (real valued) polynomials on $E$. Assume, we want to determine whether or not we have a local minimum of $f$ at $X_0$ on

$$G = \{X \in E : g_i(X) \geq 0 \text{ for } i = 1, \dots, k\}. \tag{3.24}$$

For simplicity, we further assume $(\operatorname{grad} f)(X_0) \neq 0$ and $g_i(X_0) = 0$, as well as $(\operatorname{grad} g_i)(X_0) \neq 0$, for $i = 1, \dots, k$. Then, in a sufficiently small neighborhood of $X_0$, the polynomials $f$ and $g_i$ can be approximated arbitrarily close by corresponding affine functions. For example $f$ is approximated by the beginning of its *Taylor series*

$$f(X_0) + \langle(\operatorname{grad} f)(X_0), X - X_0\rangle.$$

From this one easily derives the following well known criterion (cf. for example [**146**]) for an isolated local minimum of $f$ at $X_0$, depending on the normal cone

$$\mathcal{V}(X_0) = \operatorname{cone}\{(\operatorname{grad} g_i)(X_0) : i = 1, \dots, k\}.$$

The function $f$ attains an isolated local minimum on $G$, if

$$(\operatorname{grad} f)(X_0) \in \operatorname{int} \mathcal{V}(X_0), \tag{3.25}$$

and $f$ does not attain a local minimum, if

$$(\operatorname{grad} f)(X_0) \notin \mathcal{V}(X_0). \tag{3.26}$$

The behavior in case of $(\operatorname{grad} f)(X_0) \in \operatorname{bd} \operatorname{cone} \mathcal{V}(X_0)$ depends on the involved functions $f$ and $g_i$ and has to be treated, depending on the specific problem.

In case of the lattice sphere packing problem, we have $E = \mathcal{S}^d$, $f = \det^{1/d}$ and for $Q_0 \in \mathcal{S}^d_{>0}$ we set $g_i(Q) = Q[\boldsymbol{v}_i] - \lambda(Q_0)$ with $(\operatorname{grad} g_i)(Q) = \boldsymbol{v}_i \boldsymbol{v}_i^t$ for each pair $\pm\boldsymbol{v}_i$ in $\operatorname{Min} Q_0$. By Theorem 3.9 we have a local minimum of $f(Q) = (\det Q)^{1/d}$ at $Q_0$ on $G$ (as in (3.24)) if and only if $Q_0$ is perfect and eutactic, respectively if $\mathcal{V}(Q_0)$ is full-dimensional and $(\operatorname{grad} f)(Q_0) \in \operatorname{int} \mathcal{V}(Q_0)$. Here, $(\operatorname{grad} f)(Q_0)$ is a positive multiple of $Q_0^{-1}$. Thus in this special case we do not have a local minimum of $f$ in case $(\operatorname{grad} f)(Q_0) \in \operatorname{bd} \operatorname{cone} \mathcal{V}(Q_0)$.

Let us consider the case of periodic sets, hence of $E = \mathcal{S}^{d,m}$ with $m > 1$. We want to know if a periodic form $X_0 \in \mathcal{S}^{d,m}_{>0}$ attains a local minimum of $f = \det^{1/d}$. We may assume $\lambda(X_0) > 0$. The set $\operatorname{Min} X_0$ is finite and moreover, for $X = (Q, \boldsymbol{t})$ in a small neighborhood of $X_0 = (Q_0, \boldsymbol{t}^0)$, every $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ corresponds to a $\boldsymbol{t}_i^0 - \boldsymbol{t}_j^0 - \boldsymbol{v} \in \operatorname{Min} X_0$. Thus locally at $X_0$, the generalized Ryshkov set $\mathcal{P}_{m,\lambda}$ is given by the basic closed semialgebraic set $G$ defined by the inequalities

$p_{i,j,\boldsymbol{v}}(X) - \lambda(X_0) \geq 0$, one for each pair $\pm(\boldsymbol{t}_i^0 - \boldsymbol{t}_j^0 - \boldsymbol{v})$ in $\operatorname{Min} X_0$. As explained in Section 3.2.2, we may assume $1 \leq i \leq j \leq m$ and $\boldsymbol{t}_j^0 = \boldsymbol{0}$ if $j = m$. An elementary calculation yields

$$(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = (\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}, \dots, \boldsymbol{0}, 2Q\boldsymbol{w}, \boldsymbol{0}, \dots, \boldsymbol{0}, -2Q\boldsymbol{w}, \boldsymbol{0}, \dots, \boldsymbol{0}), \tag{3.27}$$

where we set $X = (Q, \boldsymbol{t})$ and use $\boldsymbol{w}$ to abbreviate $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}$. This is to be understood as a vector in $\mathcal{S}^{d,m} = \mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}$, with its "$\mathcal{S}^d$-component" being the rank-1 form $\boldsymbol{w}\boldsymbol{w}^t$ and its "translational-component" containing the zero-vector $\boldsymbol{0}$ in all, but the $i$th and $j$th column. In case $j = m$, the $j$th column is omitted and in case $i = j$ the corresponding column is $\boldsymbol{0}$. For $(\operatorname{grad} f)(X)$ we obtain a positive multiple of $(Q^{-1}, \boldsymbol{0})$.

**3.3.2. A sufficient condition for local $m$-periodic sphere packing optima.** Generalizing the notion of perfectness, we say a periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$ (and a corresponding periodic set represented by $X$) is *$m$-perfect* if the *generalized Voronoi domain*

$$\mathcal{V}(X) = \operatorname{cone}\{(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) : \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X \text{ for some } \boldsymbol{v} \in \mathbb{Z}^d\} \tag{3.28}$$

is full dimensional, that is, if $\dim \mathcal{V}(X) = \dim \mathcal{S}^{d,m} = \binom{d+1}{2} + (m-1)d$. Generalizing the notion of eutaxy, we say that $X$ (and a corresponding periodic set) is *$m$-eutactic* if

$$(Q^{-1}, \boldsymbol{0}) \in \operatorname{relint} \mathcal{V}(X).$$

So the general discussion in Section 3.3.1 yields the following sufficient condition for a periodic form $X$ to be *isolated $m$-extreme*, that is, for $X$ having the property that any sufficiently small change which preserves $\lambda(X)$, necessarily lowers $\delta(X)$.

Theorem 3.11. *If a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ is $m$-perfect and $m$-eutactic, then $X$ is isolated $m$-extreme.*

The theorem gives a computational tool to certify isolated $m$-extremeness of a given periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$: First, we compute $\operatorname{Min} X$ and use equation (3.27) to obtain generators of the generalized Voronoi domain $\mathcal{V}(X)$. Next, we can computationally test whether $(Q^{-1}, \boldsymbol{0})$ is in $\mathcal{V}(X)$ or not (for example by solving a linear program similar to (3.14)). In case we can show $(Q^{-1}, \boldsymbol{0}) \in \operatorname{int} \mathcal{V}(X)$, the periodic form $X$ represents an isolated $m$-extreme periodic set. If we can show $(Q^{-1}, \boldsymbol{0}) \notin \mathcal{V}(X)$, the periodic form $X$ does not represent an $m$-extreme periodic set. In this situation, we can even find a "direction" $N \in \mathcal{S}^{d,m}$, for which we can improve the sphere packing density of the periodic form $X$, that is, such that $\delta(X + \epsilon N) > \delta(X)$ for all sufficiently small $\epsilon > 0$.

Remark 3.12. Let $X \in \mathcal{S}^{d,m}_{>0}$ with $(Q^{-1}, \boldsymbol{0}) \notin \mathcal{V}(X)$. Then we can improve the sphere packing density of $X$ in direction $N$ given by the nearest point to $-(Q^{-1}, \boldsymbol{0})$ in the polyhedral cone

$$\mathcal{P}(X) = \{N \in \mathcal{S}^{d,m} : \langle V, N \rangle \geq 0 \text{ for all } V \in \mathcal{V}(X)\}. \tag{3.29}$$

Note that the cone $\mathcal{P}(X)$ is dual to the generalized Voronoi domain $\mathcal{V}(X)$ and (added to $X$) gives locally a linear approximation of the generalized Ryshkov set $\mathcal{P}_{m,\lambda}$. It generalizes the polyhedral cone (3.11) we used to describe Voronoi's algorithm.

**3.3.3. Fluid diamond packings.** For general $m$ we are confronted with a difficulty which does not show up in the lattice case $m = 1$: There may be non-isolated $m$-extreme sets, which are not $m$-perfect. The *fluid diamond packings* in dimension 9, described by Conway and Sloane in [**67**], give such an example.

EXAMPLE 3.13. The *root lattice* $\mathsf{D}_d$ can be defined by

$$\mathsf{D}_d = \{\boldsymbol{x} \in \mathbb{Z}^d : \sum_{i=1}^{d} x_i \equiv 0 \mod 2\}.$$

The *fluid diamond packings* are 2-periodic sets

$$\mathsf{D}_9\langle \boldsymbol{t} \rangle = \mathsf{D}_9 \cup (\mathsf{D}_9 + \boldsymbol{t})$$

with $\boldsymbol{t} \in \mathbb{R}^9$ such that the minimal distance among elements is equal to the minimum distance $\sqrt{2}$ of $\mathsf{D}_9$ itself. We may choose for example $\boldsymbol{t}_\alpha = (\frac{1}{2}, \ldots, \frac{1}{2}, \alpha)^t$ with any $\alpha \in \mathbb{R}$. For integers $\alpha$ we obtain the densest known packing lattice $\Lambda_9 = \mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$ in dimension 9, showing that it is part of a family of uncountably many, equally dense 2-periodic sets.

The sets $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$ give examples of non-isolated 2-extreme sets, which are 2-eutactic, but not 2-perfect. In order to see this, let us consider a representation $X_\alpha \in \mathcal{S}^{9,2}_{>0}$ for $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$. We choose a basis $A$ of $\mathsf{D}_9$. Then $X_\alpha = (Q, A^{-1}\boldsymbol{t}_\alpha)$ with $Q = A^t A$ is a representation of $\mathsf{D}_9\langle \boldsymbol{t}_\alpha \rangle$.

For non-integral $\alpha$ we find $\operatorname{Min} X_\alpha = \operatorname{Min} Q$ (using `MAGMA` for example). It follows (for example by Lemma 3.16) that $X_\alpha$ is 2-eutactic, but not 2-perfect. For integral $\alpha$ we find

$$\operatorname{Min} X_\alpha = \operatorname{Min} Q \cup \{(x_1, \ldots, x_8, 0)^t \in \{0,1\}^9 : \sum_{i=1}^{8} x_i \equiv 0 \mod 2\}.$$

Thus the vectors in $\operatorname{Min} X_\alpha \setminus \operatorname{Min} Q$ span only an 8-dimensional space. Therefore, $X_\alpha$ not 2-perfect. Nevertheless, a corresponding calculation shows that $X_\alpha$ is 2-eutactic as in the case of non-integral $\alpha$.

In order to see that $X_\alpha$ is non-isolated 2-extreme, we can apply Proposition 3.14 below. One easily checks that in case of integral $\alpha$ (hence for the lattice $\Lambda_9$) we have only one degree of freedom for a local change of $\boldsymbol{t}_\alpha$ giving an equally dense sphere packing. In case of non-integral $\alpha$ we have nine degrees of freedom for such a modification.

□

In the terminology of Section 3.3.2, non-isolated $m$-extreme sets as in this example can only occur for periodic forms $X \in \mathcal{S}^{d,m}_{>0}$ if $(Q^{-1}, \mathbf{0}) \in \operatorname{bd} \mathcal{V}(X)$ (which is for example always the case when $X$ is $m$-eutactic, but not $m$-perfect). In this case, it is in general not clear what an infinitesimal change of $X$ in a direction $N$ leads to (already assuming it is orthogonal to $(Q^{-1}, \mathbf{0})$ as well as in the boundary of the set $\mathcal{P}(X)$ in (3.29)). If $\mathcal{F}(X)$ denotes the unique face of $\mathcal{V}(X)$ containing $(Q^{-1}, \mathbf{0})$ in its relative interior, then this "set of uncertainty" is equal to the face of $\mathcal{P}(X)$ dual to $\mathcal{F}(X)$, that is, equal to

$$\mathcal{U}(X) = \{N \in \mathcal{P}(X) : \langle V, N \rangle = 0 \text{ for all } V \in \mathcal{F}(X)\}. \tag{3.30}$$

Or in other words, the set $\mathcal{U}(X)$ is the intersection of $\mathcal{P}(X)$ with the hyperplane orthogonal to $(Q^{-1}, \mathbf{0})$. Note that it is possible to determine $\mathcal{F}(X)$ (and hence a description of $\mathcal{U}(X)$ by linear inequalities) computationally, using linear programming techniques.

**3.3.4. Purely translational changes.** Below we give an additional sufficient condition for $m$-extremeness. For this we consider the case when all directions in $\mathcal{U}(X)$ are "purely translational changes" $N = (0, \boldsymbol{t}^N) \in \mathcal{S}^{d,m}$. A vivid interpretation of a purely translational change can be given by thinking of the corresponding modification of a periodic sphere packing. The spheres of a lattice translate are jointly moved. If in such a local change all contacts among spheres are lost, we can increase their radius and obtain a new sphere packing with larger density. If some contacts among spheres are preserved however, the sphere packing density remains the same. Latter case is captured in the following proposition, which gives an easily testable criterion for $m$-extremeness. We apply this proposition in Section 3.3.5, where we consider potential local improvements of best known packing lattices to periodic non-lattice sets.

PROPOSITION 3.14. *For a periodic form* $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$ *with* $(Q^{-1}, \mathbf{0}) \in \operatorname{bd} \mathcal{V}(X)$*, let* $\mathcal{U}(X)$ *be contained in*

$$\{(\mathbf{0}, \boldsymbol{t}^N) \in \mathcal{S}^{d,m} : \boldsymbol{t}_i^N = \boldsymbol{t}_j^N \textit{ for at least one } \boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X \textit{ with } \boldsymbol{v} \in \mathbb{Z}^d\}.$$

*Then $X$ is (possibly non-isolated) $m$-extreme.*

Note, if $X$ is $m$-eutactic (possibly not $m$-perfect), the set $\mathcal{U}(X)$ is the *orthogonal complement* $\mathcal{V}(X)^\perp$ of the linear hull of $\mathcal{V}(X)$. Note also that the proposition includes in particular the special case where some $\boldsymbol{v} \in \mathbb{Z}^d$ are in $\operatorname{Min} X$ (and therefore $\boldsymbol{t}_i = \boldsymbol{t}_j = \mathbf{0}$ for $i = j = m$). This situation occurs for the 2-periodic, fluid diamond packings of Section 3.3.3.

From the sphere packing interpretation of Proposition 3.14 its assertion is clear. Nevertheless, we give a proof below, based on a local analysis in $\mathcal{S}^{d,m}_{>0}$. More than actually needed for the proof, we analyze how $\delta$ changes locally at a periodic form $X \in \mathcal{S}^{d,m}_{>0}$ in a direction $N \in \mathcal{U}(X)$. As a byproduct, we obtain tools allowing a computational analysis of possible local optimality for a given periodic form, not necessarily covered by the proposition. These can for example be used in a numerical search for good periodic sphere packings.

PROOF OF PROPOSITION 3.14. The generalized Voronoi domain $\mathcal{V}(X)$ is spanned by gradients $(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)$ (as given in (3.27)), one for each pair of vectors $\pm\boldsymbol{w} \in \operatorname{Min} X$. The assumption that a direction $N = (Q^N, \boldsymbol{t}^N)$ is in $\mathcal{U}(X)$ for a periodic form $X = (Q, \boldsymbol{t})$, implies $\langle Q^{-1}, Q^N \rangle = 0$. Moreover, for the unique maximal face $\mathcal{F}(X)$ of $\mathcal{V}(X)$ with $(Q^{-1}, \mathbf{0}) \in \operatorname{relint} \mathcal{F}(X)$, the condition that $N$ is orthogonal to some $(\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)$ in $\mathcal{F}(X)$ translates into

$$\langle (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X), N \rangle = Q^N[\boldsymbol{w}] + 2(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q \boldsymbol{w} = 0, \tag{3.31}$$

with $\boldsymbol{w} = (\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v})$. Recall that in the special case $i = j$ (and for $m = 1$ anyway) $p_{i,j,\boldsymbol{v}}$ is linear and (3.31) reduces to the condition $Q^N[\boldsymbol{w}] = 0$; if then $N$ satisfies this linear condition, $p_{i,j,\boldsymbol{v}}(X + \epsilon N)$ is a constant function in $\epsilon$.

In case $p_{i,j,\boldsymbol{v}}(X + \epsilon N)$ is a cubic polynomial in $\epsilon$ we need to use higher order information in order to judge its behavior. An elementary calculation yields for the

*Hessian*

$$(\operatorname{hess} p_{i,j,\boldsymbol{v}})(X)[N] = 2Q[\boldsymbol{t}_i^N - \boldsymbol{t}_j^N] + 4(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q^N \boldsymbol{w}. \tag{3.32}$$

Now how does $\delta$ change at $X$ in direction $N$, assuming it is in the set of uncertainty $\mathcal{U}(X)$? Among the polynomials $p_{i,j,\boldsymbol{v}}$ with $N$ satisfying (3.31), the fastest decreasing polynomial in direction $N$ determines $\lambda(X+\epsilon N)$ for small enough $\epsilon$. Thus for the local change of $\delta$ in direction $N$, we may restrict our attention to a polynomial $p_{i,j,\boldsymbol{v}}$ with the smallest value (3.32) of its Hessian.

By Theorem 2.3 we know that $\det^{1/d}$ decreases strictly at $X$ in a direction $N \in \mathcal{U}(X)$ if and only if $Q^N \neq 0$.

In case of a purely translational change with $Q^N = 0$, the function $\det^{1/d}$ remains constant. On the other hand, because of (3.32) and since $Q$ is positive definite, we have $(\operatorname{hess} p_{i,j,\boldsymbol{v}})(X)[N] \geq 0$, with equality if and only if $\boldsymbol{t}_i^N - \boldsymbol{t}_j^N = \boldsymbol{0}$. Latter implies that $p_{i,j,\boldsymbol{v}}(X+\epsilon N)$ is a constant function of $\epsilon$. Thus for purely translational changes $N = (0, \boldsymbol{t}^N) \in \mathcal{U}(X)$, the density function $\delta(X+\epsilon N)$ is constant for small enough $\epsilon \geq 0$, if $\boldsymbol{t}_i^N = \boldsymbol{t}_j^N$ for some pair $(i,j)$ with $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ (for a suitable $\boldsymbol{v} \in \mathbb{Z}^d$). This proves the proposition. □

Note that our argumentation in the proof also shows that $\delta(X+\epsilon N)$ increases for small $\epsilon > 0$, in case of a purely translational change $N = (0, \boldsymbol{t}^N) \in \mathcal{U}(X)$ with $\boldsymbol{t}_i^N \neq \boldsymbol{t}_j^N$ for all pairs $(i,j)$ with $\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v} \in \operatorname{Min} X$ (for some $\boldsymbol{v} \in \mathbb{Z}^d$). This case corresponds to a modification of a periodic sphere packing, in which all contacts among spheres are lost.

**3.3.5. Periodic extreme sets.** A given periodic set has many representations by periodic forms, in spaces $\mathcal{S}^{d,m}_{>0}$ with varying $m$. For example, by choosing some sublattice of $\mathbb{Z}^d$, we can add additional translational parts.

Now it could happen that a periodic set $\Lambda$ with a given representation $X \in \mathcal{S}^{d,m}_{>0}$ is $m$-extreme, whereas a second representation $X' \in \mathcal{S}^{d,m'}$ is not $m'$-extreme. However, it may also happen that the packing density of no representation of $\Lambda$ can locally be improved.

DEFINITION 3.15. A periodic set is *periodic extreme*, if it is $m$-extreme for all possible representations $X \in \mathcal{S}^{d,m}_{>0}$.

The main result of this section is Theorem 3.17 below, giving a sufficient condition for a lattice to be periodic extreme. For its statement we need the notion of *strong eutaxy* for lattices, respectively PQFs: A form $Q \in \mathcal{S}^d_{>0}$ (and a corresponding lattice) is called *strongly eutactic*, if

$$Q^{-1} = \alpha \sum_{\boldsymbol{x} \in \operatorname{Min} Q} \boldsymbol{x}\boldsymbol{x}^t \tag{3.33}$$

for some $\alpha > 0$, hence if the coefficients in the eutaxy condition (3.13) are all equal. A PQF $Q$ is strongly eutactic, if and only if the vectors in $\operatorname{Min} Q$ form a *spherical 2-design* with respect to $Q$ (see Section 5.5.3).

LEMMA 3.16. *Any representation $X \in \mathcal{S}^{d,m}_{>0}$ of a strongly eutactic lattice (respectively PQF) is $m$-eutactic.*

Proof. Let $Q \in \mathcal{S}^d_{>0}$ be strongly eutactic, satisfying (3.33) for some $\alpha > 0$. Let $X = (Q^X, \boldsymbol{t}^X) \in \mathcal{S}^{d,m}_{>0}$ be some representation of a strongly eutactic PQF $Q$, e.g. with $m > 1$.

For a fixed $\boldsymbol{w} \in \operatorname{Min} X$ we define an abstract graph, whose vertices are the indices in $\{1,\dots,m\}$. Two vertices $i$ and $j$ are connected by an edge, whenever there is some $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{w} = \boldsymbol{t}^X_i - \boldsymbol{t}^X_j - \boldsymbol{v}$. Since the periodic form $X$ represents a lattice, we find that the graph is a disjoint union of cycles. Or, in other words, $\boldsymbol{w}$ induces a partition $(I_1,\dots,I_k)$ of $\{1,\dots,m\}$.

Let $I$ be an index set of this partition (containing the indices of a fixed cycle of the defined graph). Summing over all triples $(i,j,\boldsymbol{v})$ with $i,j \in I$ and $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{w} = \boldsymbol{t}^X_i - \boldsymbol{t}^X_j - \boldsymbol{v} \in \operatorname{Min} X$, we find (using (3.27)):

$$\sum_{\substack{(i,j,\boldsymbol{v}) \in I^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}^X_i - \boldsymbol{t}^X_j - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = 2|I|(\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}).$$

The factor 2 comes from the symmetry $\operatorname{grad} p_{i,j,\boldsymbol{v}} = \operatorname{grad} p_{j,i,-\boldsymbol{v}}$. Summation over all index sets $I$ of the partition yields

$$\sum_{\substack{(i,j,\boldsymbol{v}) \in \{1,\dots,m\}^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}^X_i - \boldsymbol{t}^X_j - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) = 2m(\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0}). \tag{3.34}$$

As a consequence we find by the strong eutaxy condition (3.33) that

$$(Q^{-1}, \boldsymbol{0}) \; = \; (\alpha/2m) \sum_{\substack{(i,j,\boldsymbol{v}) \in \{1,\dots,m\}^2 \times \mathbb{Z}^d: \\ \boldsymbol{w} = \boldsymbol{t}^X_i - \boldsymbol{t}^X_j - \boldsymbol{v} \in \operatorname{Min} X}} (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X),$$

with a suitable $\alpha > 0$. Thus $X$ is $m$-eutactic. □

Not all PQFs (or lattices) which are strongly eutactic have to be perfect. But if a strongly eutactic PQF is in addition also perfect, then the following theorem shows that this is sufficient for it to be periodic extreme. Note that this applies in particular to so called *strongly perfect lattices and PQFs* (see [**187**], [**174**]).

Theorem 3.17. *A perfect, strongly eutactic lattice (respectively PQF) is periodic extreme.*

Proof. Let $Q \in \mathcal{S}^d_{>0}$ be perfect and strongly eutactic. Hence the vectors in $\operatorname{Min} Q$ span $\mathbb{R}^d$ (by Proposition 3.3) and satisfy (3.33) for some $\alpha > 0$. Let $X = (Q^X, \boldsymbol{t}^X) \in \mathcal{S}^{d,m}_{>0}$ be a representation of $Q$, e.g. with $m > 1$. By Lemma 3.16, $X$ is $m$-eutactic. If $X$ is $m$-perfect as well, we know by Theorem 3.11 that $X$ is also $m$-extreme.

So let us assume that $X$ is not $m$-perfect, hence the generalized Voronoi domain $\mathcal{V}(X)$ is not full dimensional. We want to apply Proposition 3.14. For this we choose

$$N = (Q^N, \boldsymbol{t}^N) \in \mathcal{U}(X) = \mathcal{V}(X)^\perp \quad \text{with} \quad N \neq 0.$$

(Recall the definition of $\mathcal{U}(X)$ from (3.30) and that $\mathcal{U}(X) = \mathcal{V}(X)^\perp$ in case $X$ is $m$-eutactic.) By this assumption we have in particular

$$\langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X) \rangle = 0$$

for all triples $(i, j, \boldsymbol{v})$ with $\boldsymbol{w} = \boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X$. Using equation (3.34), which we obtained in the proof of Lemma 3.16, we get $\langle N, (\boldsymbol{w}\boldsymbol{w}^t, \boldsymbol{0})\rangle = Q^N[\boldsymbol{w}] = 0$ for every fixed $\boldsymbol{w} \in \operatorname{Min} X$.

By Proposition 3.3 there exist $d$ linear independent $\boldsymbol{w}$ in $\operatorname{Min} X$, which implies $Q^N = 0$. Using (3.31), we obtain

$$0 = \langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)\rangle = 2(\boldsymbol{t}_i^N - \boldsymbol{t}_j^N)^t Q\boldsymbol{w}. \tag{3.35}$$

In case $\boldsymbol{t}_i^N - \boldsymbol{t}_j^N = \boldsymbol{0}$ for some pair $(i, j)$ we can apply Proposition 3.14. Note that this includes in particular the case $i = j = m$ ($\boldsymbol{t}_i^N = \boldsymbol{t}_j^N = \boldsymbol{0}$) if $\boldsymbol{v} \in \mathbb{Z}^d \cap \operatorname{Min} X$. So we may assume that such $\boldsymbol{v}$ do not exist.

We define an abstract graph with vertices in $\{1, \ldots, m\}$: $(i, j)$ is an edge, whenever there is some $\boldsymbol{v} \in \mathbb{Z}^d$ such that $\boldsymbol{t}_i^X - \boldsymbol{t}_j^X - \boldsymbol{v} \in \operatorname{Min} X$. Let $I$ be the set of vertices (indices) $i$, connected by a path with $m$. Then by the assumption $\mathbb{Z}^d \cap \operatorname{Min} X = \emptyset$ the cardinality $|I|$ of $I$ is greater than 1. Since there exist $d$ linearly independent vectors in $\operatorname{Min} X$ (by Proposition 3.3), the linear equations $\langle N, (\operatorname{grad} p_{i,j,\boldsymbol{v}})(X)\rangle = 0$ with $i, j \in I$ have rank $d(|I| - 1)$. Thus by (3.35) and $\boldsymbol{t}_m^N = \boldsymbol{0}$, we deduce $\boldsymbol{t}_i^N = \boldsymbol{0}$ for all $i \in I$; hence we can again apply Proposition 3.14, proving the theorem. □

Note that the set $I$ used in the last paragraph of the proof can be a strict subset of $\{1, \ldots, m\}$. This is the case, if the graph we defined is not connected. Each connected component corresponds to a union of translates, which can jointly, locally be changed, without changing $\lambda(X)$, respectively $\delta(X)$. For example the fluid diamond packings described in Section 3.3.3 have this property (although $\Lambda_9$ is not strictly eutactic).

The root lattices $\mathsf{A}_d$, $\mathsf{D}_d$ and $\mathsf{E}_d$, as well as the Leech lattice are known to be perfect and strongly eutactic (cf. [**174**]). Thus as an immediate consequence of Theorem 3.17, we find that these lattices, which are known to solve the lattice sphere packing problem in dimensions $d \leq 8$ and $d = 24$, cannot locally be improved to a periodic non-lattice set with greater sphere packing density.

Corollary 3.18. *The lattices $\mathsf{A}_d$, for $d \geq 2$, $\mathsf{D}_d$, for $d \geq 3$, and $\mathsf{E}_d$ for $d = 6, 7, 8$, as well as the Leech lattice are periodic extreme.*

A result similar to Corollary 3.18 for the root lattices $\mathsf{A}_d$ and $\mathsf{D}_d$ was obtained by Bezdek, Bezdek and Connelly [**39**, Remark 2.9.1]. Their method is different and based on *Delone subdivisions* (see Section 4.1.1). At this point it is not clear whether or not their method can be used to obtain a more general result similar to Theorem 3.17.

We also checked whether or not Theorem 3.17 can be applied to other dimensions $d$ below 24. For these dimensions the so-called *laminated lattices* $\Lambda_d$ and *sections $K_d$ of the Leech lattice* give the densest known lattice sphere packings. The lattices $K_d$ are different from $\Lambda_d$ (and at the same time give the densest known lattice sphere packings) only in dimensions $d = 11, 12, 13$. For these $d$, the lattice $K_d$ is strongly eutactic only for $d = 12$, where $K_d$ is also known as *Coxeter-Todd lattice*. The laminated lattices $\Lambda_d$ give the densest known packing lattices in dimensions $d = 9, 10$ and $d = 14, \ldots, 24$ (for $d = 18, \ldots, 24$ they coincide with $K_d$). Among those values for $d$, the laminated lattices $\Lambda_d$ are strongly eutactic if and only if $d = 15, 16$ or $d \geq 20$. Concluding, we cannot exclude that densest known lattice sphere packings in dimensions $d \in \{9, 10, 11, 13, 14, 17, 18, 19\}$ can locally be improved to better periodic sphere packings. Further analysis is required here.

## 3.4. Equivariant extension and $T$-perfect forms

**3.4.1. Automorphism groups of periodic sets.** In Section 1.2, we identified the set of all $d$-dimensional lattices with the quotient space $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$. A similar identification for periodic sets is possible, as we shall see in this section. Let a periodic set be represented by a periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$, with $Q \in \mathcal{S}^d_{>0}$ and with $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$. The translational part $\boldsymbol{t}$ defines the standard periodic set

$$\Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d, \tag{3.36}$$

where we set $\boldsymbol{t}_m = \mathbf{0}$ again. We define the *group of automorphisms of* $X$ by

$$\operatorname{Aut} X = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U\Lambda_{\boldsymbol{t}} = \Lambda_{\boldsymbol{t}} \text{ and } U^t Q U = Q\}. \tag{3.37}$$

By definition it is a subgroup of the *automorphism group*

$$\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z}) = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U\Lambda_{\boldsymbol{t}} = \Lambda_{\boldsymbol{t}}\} \tag{3.38}$$

*of the standard periodic set* $\Lambda_{\boldsymbol{t}}$. In case $m = 1$, we simply mean $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z}) = \mathsf{GL}_d(\mathbb{Z})$.

In case $X$ represents a *rational periodic set* (with rational translational part $\boldsymbol{t}$), $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$ is a *congruence subgroup*, that is, a group containing a *principal congruence subgroup of level* $N \in \mathbb{N}$

$$\Gamma_d(N) = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U \equiv \mathrm{id}_d \mod N\}.$$

In fact, the following proposition holds:

Proposition 3.19. *Let $d, m \geq 2$. Then the group $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$ is a congruence subgroup and has a finite* index $|\mathsf{GL}_d(\mathbb{Z}) : \mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})|$ *in $\mathsf{GL}_d(\mathbb{Z})$ if and only if $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$ is rational.*

Proof. In the case of rational $\boldsymbol{t}$, the group $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$ is a congruence subgroup because it contains the principal congruence subgroup $\Gamma_d(N)$ with $N$ being the least common multiple of the denominators in $\boldsymbol{t}$. In general, congruence subgroups have a finite index in $\mathsf{GL}_d(\mathbb{Z})$, since $\Gamma_d(N)$ is the kernel of the canonical homomorphism $\mathsf{GL}_d(\mathbb{Z}) \to \mathsf{GL}_d(\mathbb{Z}/N\mathbb{Z})$ (thus the index $|\mathsf{GL}_d(\mathbb{Z}) : \Gamma_d(N)|$ of $\Gamma_d(N)$ in $\mathsf{GL}_d(\mathbb{Z})$ is finite).

It remains to show that the index $|\mathsf{GL}_d(\mathbb{Z}) : \mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})|$ is not finite in case of non-rational $\boldsymbol{t}$. For this it is sufficient to show that the set

$$\{U\Lambda_{\boldsymbol{t}} : U \in \mathsf{GL}_d(\mathbb{Z})\} \tag{3.39}$$

is infinite for non-rational $\boldsymbol{t}$. Assume w.l.o.g. that $t_{11}$ is irrational. Then

$$\{\ kt_{11} \bmod 1\ :\ k \in \mathbb{Z}\ \}$$

is dense in $[0, 1)$ (and even "equidistributed" by a classical result of Weyl). Now for any $k \in \mathbb{Z}$ it is possible to construct (infinitely many) matrices $U \in \mathsf{GL}_d(\mathbb{Z})$ with $u_{11} = k$, $u_{12} = 1$ and $u_{1j} = 0$ for $j > 2$ (since this first row vector of $U$ can be completed to a basis of $\mathbb{Z}^d$). The same applies to the $i$-th row (for any $i \in \{1, \ldots, d\}$) and we find that $\{(U\boldsymbol{t}_1)_i \bmod 1 : U \in \mathsf{GL}_d(\mathbb{Z})\}$ is infinite, showing that (3.39) must be infinite as well. ∎

Congruence subgroups are well studied objects and play an important role for example in the theory of automorphic forms. As seen in the proof above, any congruence subgroup has a finite index in $\mathsf{GL}_d(\mathbb{Z})$.

By the solution of the so-called "congruence subgroup problem", which was independently found by Bass, Lazard, Serre [**29**] and Mennicke [**178**] (cf. [**30**]), the converse is also true for all $d \geq 3$. That is, any subgroup $G$ of $\mathsf{GL}_d(\mathbb{Z})$ having finite index $|\mathsf{GL}_d(\mathbb{Z}) : G|$ is a congruence subgroup (a group containing some principal congruence subgroup of level $N$ as a subgroup). Note that for $d = 2$ this does not hold, as already observed by Klein in the 19th century. For example, there is a surjective homomorphism $\mathsf{GL}_2(\mathbb{Z}) \to A_5$ (where here, $A_5$ denotes the *alternating group* on 5 elements) whose kernel is not a congruence subgroup. Let us also note that $\Gamma_d(N)$ is *torsion free* (has no elements of finite order) for all $N \geq 3$ and has only 2-torsion elements for $T = 2$, i.e. $-\operatorname{id}_d \in \Gamma_d(2)$. However, a given congruence subgroup as for example the group $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with rational $\boldsymbol{t}$ may contain a larger finite subgroup.

EXAMPLE 3.20. Consider $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{2,2}_{>0}$ with $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\boldsymbol{t} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ is a *semidirect product* of $\Gamma_2(3)$ and the group

$$\operatorname{Aut} X = \left\{ \operatorname{id}_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

**3.4.2. Periodic sets with fixed translational part.** Given a fixed, translational part $\boldsymbol{t} \in \mathbb{R}^{d \times (m-1)}$, respectively a corresponding standard periodic set (3.36), we define its *Ryshkov polyhedron* as

$$\mathcal{P}_{\lambda,\boldsymbol{t}} = \{Q \in \mathcal{S}^d_{>0} : (Q, \boldsymbol{t}) \in \mathcal{P}_{m,\lambda}\} \subset \mathcal{S}^d_{>0}.$$

It may be viewed as the intersection of $\mathcal{P}_{m,\lambda}$ with the affine subspace in $\mathcal{S}^{d,m}_{>0}$ having the translational part $\boldsymbol{t}$ fixed. Note that a choice of $\boldsymbol{t}$ that forces $\lambda(X) = 0$ for every periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$, gives $\mathcal{P}_{\lambda,\boldsymbol{t}} = \emptyset$, by our assumption $\lambda > 0$. So we may a priori exclude this case from further considerations.

Our first observation is a generalization of Theorem 3.1:

THEOREM 3.21. *Let $\lambda > 0$, $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d \times (m-1)}$. Then $\mathcal{P}_{\lambda,\boldsymbol{t}}$ is a locally finite polyhedron.*

As in the proof of Theorem 3.1 one shows that

$$\mathcal{P}_{\lambda,\boldsymbol{t}} \cap \{Q \in \mathcal{S}^d : \operatorname{trace} Q \leq C\} \tag{3.40}$$

is a polytope for every constant $C$. We omit the proof here, since the proof of Theorem 3.1 immediately generalizes to this case.

In analogy to the classical theory, we call the vertices $Q$ of $\mathcal{P}_{\lambda,\boldsymbol{t}}$ (and the corresponding periodic form $X = (Q, \boldsymbol{t})$) ***t**-perfect*. By Theorem 2.3 we know that among periodic forms $X = (Q, \boldsymbol{t})$ with a fixed translational part $\boldsymbol{t}$, the ***t**-extreme* ones (attaining a local maximum of $\delta$) have to be $\boldsymbol{t}$-perfect. We call a periodic form $X = (Q, \boldsymbol{t})$, respectively the PQF $Q$, ***t**-eutactic*, if $Q^{-1}$ is contained in the relative interior of the normal cone of $\mathcal{P}_{\lambda(Q),\boldsymbol{t}}$ at $Q$. As in the classical theory, we have a complete characterization of $\boldsymbol{t}$-extreme periodic sets, which follows from the general discussion in Section 3.3.1.

THEOREM 3.22. *A periodic form $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$ (respectively $Q \in \mathcal{S}^d_{>0}$) is $\boldsymbol{t}$-extreme if and only if it is $\boldsymbol{t}$-perfect and $\boldsymbol{t}$-eutactic.*

**3.4.3. Extending Voronoi's algorithm.** The automorphism group $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ of the standard periodic set $\Lambda_{\boldsymbol{t}}$ (see (3.36)) acts on $\mathcal{P}_{\lambda,\boldsymbol{t}}$, and in particular on its vertices ($\boldsymbol{t}$-perfect forms) and edges. Thus Voronoi's algorithm generalizes to a graph traversal search that determines all $\boldsymbol{t}$-perfect forms up to $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalence. Here, two PQFs $Q$ and $Q'$ are called $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-*equivalent*, if there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with $Q' = U^t Q U$.

The following theorem shows that the generalization of Voronoi's algorithm is actually an algorithm (finishes in finitely many steps) for rational $\boldsymbol{t}$. We include the classical case $m = 1$, for which one has to read $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ as $\mathsf{GL}_d(\mathbb{Z})$ and $\boldsymbol{t}$-perfectness as perfectness.

THEOREM 3.23. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$. Then up to scaling and $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalence, there exist only finitely many $\boldsymbol{t}$-perfect forms.*

PROOF. As in the proof of Theorem 3.4, we use our observation that (3.40) is a polytope. Then the finiteness for rational $\boldsymbol{t}$ is guaranteed if we show for fixed $\lambda > 0$ that every $\boldsymbol{t}$-perfect form $Q$ with $\lambda((Q, \boldsymbol{t})) = \lambda$ is $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent to a PQF with trace smaller than some constant, depending only on $d$, $\lambda$ and $\boldsymbol{t}$.

We choose $N$ such that the congruence subgroup $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ contains the principal congruence subgroup $\Gamma_d(N)$. Choose for $N$ for example the least common multiple of denominators in $\boldsymbol{t}$. Application of Hermite's Theorem 1.1, yields a $\tilde{U} \in \mathsf{GL}_d(\mathbb{Z})$ such that $\tilde{Q} = \tilde{U}^t Q \tilde{U}$ satisfies

$$\prod_{i=1}^{d} \tilde{q}_{ii} \leq \tilde{C} \cdot \det \tilde{Q} \tag{3.41}$$

for $\tilde{C} = \left(\frac{4}{3}\right)^{d(d-1)/2}$. The determinant $\det \tilde{Q}$ can be bounded by $\lambda^d N^{2d}$, using Hadamard's inequality and the fact that (by the $\boldsymbol{t}$-perfectness of $Q$) there exist $d$ linearly independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d \in \mathbb{Z}^d$ with $\tilde{Q}[\boldsymbol{a}_i] = N^2 \tilde{Q}[\boldsymbol{a}_i/N] = N^2\lambda$. Note that the existence of these $d$ linearly independent vectors follows for the same reasons as in Proposition 3.3. Because of $\tilde{q}_{ii} = \tilde{Q}[\boldsymbol{e}_i] \geq \lambda$ we obtain from (3.41) a bound for the trace of $\tilde{Q}$:

$$\operatorname{trace} \tilde{Q} = \sum_{i=1}^{d} \tilde{q}_{ii} \leq \sum_{i=1}^{d} \tilde{C} N^{2d} \lambda = d\tilde{C} N^{2d} \lambda \tag{3.42}$$

If $\tilde{Q}$ and $Q$ are not already $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent, we choose the unique $\tilde{U}' \in \{0, \ldots, N-1\}^{d\times d}$ with $\tilde{U}' \equiv \tilde{U}^{-1} \bmod N$ and define $U = \tilde{U}\tilde{U}'$. Then by definition, $U \equiv \mathrm{id}_d \bmod N$, hence $U \in \Gamma_d(N) \subseteq \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ and

$$Q' = U^t Q U = \tilde{U}'^t \tilde{Q} \tilde{U}'$$

is $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent to $Q$. Its diagonal entries $q'_{ii} = \tilde{Q}[\tilde{u}'_i]$ (with $\tilde{u}'_i$ the $i$-th column of $\tilde{U}'$) satisfy

$$q'_{ii} \leq \sum_{k,l=1}^{d} |\tilde{q}_{kl}||\tilde{u}'_{ik}||\tilde{u}'_{il}| \leq (N-1)^2 \sum_{k,l=1}^{d} |\tilde{q}_{kl}|.$$

Using $|\tilde{q}_{kl}| \leq \frac{1}{2}(\tilde{q}_{kk} + \tilde{q}_{ll})$ (which follows from $\tilde{Q}[\boldsymbol{e}_k \pm \boldsymbol{e}_l] \geq 0$) we obtain

$$q'_{ii} \leq (N-1)^2 \sum_{k=1}^{d} \sum_{l=1}^{d} \frac{1}{2}(\tilde{q}_{kk} + \tilde{q}_{ll}) = d(N-1)^2 \operatorname{trace} \tilde{Q}. \tag{3.43}$$

Combining (3.42) and (3.43), we get the desired bound for the trace of the $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent form $Q'$ of $Q$. $\square$

We think for non-rational $\boldsymbol{t}$ it can be shown that there exist infinitely many pairwise $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent $\boldsymbol{t}$-perfect forms.

In order to run the generalization of Voronoi's algorithm in practice, we have to be able to test computationally, whether or not two PQFs $Q$ and $Q'$ are $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent. This can be achieved simply by testing for each of the finitely many $U \in \mathsf{GL}_d(\mathbb{Z})$ with $Q' = U^tQU$, if they belong to $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. For example, if $\boldsymbol{t}$ is rational and $N$ is the least common multiple of the denominators in $\boldsymbol{t}$, we can test if

$$UN\boldsymbol{t} \equiv N\boldsymbol{t}' \mod N$$

for some $\boldsymbol{t}'$ being a copy of $\boldsymbol{t}$ with possibly permuted columns. Similarly, we can compute the $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-*restricted automorphism group*

$$\operatorname{Aut}_{\boldsymbol{t}} Q = \operatorname{Aut}(Q, \boldsymbol{t})$$

of a PQF $Q$ (see (3.37)).

Note that we have everything necessary to practically implement Voronoi's algorithm for the enumeration of all $\boldsymbol{t}$-perfect forms (with rational $\boldsymbol{t}$). By this it becomes possible to systematically search for dense periodic sphere packings.

**3.4.4. Voronoi's algorithm with respect to a subspace.** The enumeration of all perfect forms or $\boldsymbol{t}$-perfect forms becomes practically impossible in higher dimensions due to the complexity of the Ryshkov polyhedra $\mathcal{P}_{\lambda,\boldsymbol{t}}$ (including $\mathcal{P}_\lambda$ as a special case). So one might want to consider a more restricted class of PQFs. Since the best known sphere packings all have many symmetries one natural possibility is to consider only forms which are invariant with respect to a given finite group $G \subset \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. As noted in Section 3.1.6, the set of all $G$-invariant forms is a linear subspace $T_G$ of $\mathcal{S}^d$.

Within $T_G$, in the classical $m = 1$ case, we are lead to the theory of *$G$-perfect forms* of Bergé, Martinet and Sigrist [**37**]. It generalizes to a theory of *$T$-perfect forms*, where $T \subseteq \mathcal{S}^d$ is some linear subspace (see [**174**]). Suitable linear subspaces $T$ allow systematical treatments of important classes of lattices. Examples are *Eisenstein*, *Gaussian*, and *Hurwitz quaternionic* lattices, as we explain in Section 3.4.6. For further examples, such as *patchwork lattices* and lattices with a fixed section we refer to [**174**].

Our viewpoint developed in this chapter (based on Ryshkov polyhedra) allows not only a straightforward description of these theories, but also allows us to extend these concepts easily to the periodic setting. Given a linear subspace $T \subseteq \mathcal{S}^d$ we simply consider the intersection

$$\mathcal{P}_{\lambda,\boldsymbol{t}} \cap T. \tag{3.44}$$

The vertices of the resulting locally finite polyhedra (within $T$) are called *$(T, \boldsymbol{t})$-perfect forms* (or simply *$T$-perfect forms*, if $m = 1$). In case $T = T_G$, where $G$ is a finite subgroup of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ we speak of *$(G, \boldsymbol{t})$-perfect forms* (and *$G$-perfect forms* if $m = 1$). One should be aware that in general, $(T, \boldsymbol{t})$-perfectness does not imply $\boldsymbol{t}$-perfectness. Moreover, finiteness (up to equivalence preserving $T$) may be lost (see [**145**]). However, although possibly not finishing in finitely many steps, Voronoi's algorithm generalizes to a graph traversal search of "equivalent vertices" of the locally finite polyhedra in (3.44).

Again, we have a modified notion of equivalence. Two PQFs $Q$ and $Q'$ are called $(T,\boldsymbol{t})$-*equivalent* (or $T$-*equivalent*, if $m=1$) if there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with $Q' = U^tQU$ and $U^tTU \subseteq T$. Latter condition is sufficient to guarantee equality $U^tTU = T$. If $T$ is given by a set of generating quadratic forms or inequalities, we can easily check computationally if this condition is satisfied. The same is true for the computation of $(T,\boldsymbol{t})$-*automorphisms* of $Q$, which are $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with $Q = U^tQU$ and $U^tTU \subseteq T$.

Note that $T$-perfect and $G$-perfect forms are usually defined via normal cones of faces of $\mathcal{P}_\lambda \cap T$ in $T$ (cf. [**174**], [**37**], [**144**], [**192**] and [**193**]). A face $F$ of $\mathcal{P}_\lambda$ is uniquely characterized by the set

$$\operatorname{Min} F = \{\boldsymbol{v} \in \mathbb{Z}^d : Q[\boldsymbol{v}] = \lambda \text{ for all } Q \in F\},$$

which is independent of the particular choice of $\lambda$. The normal cone of $F$ is the Voronoi domain $\operatorname{cone}\{\boldsymbol{v}\boldsymbol{v}^t : \boldsymbol{v} \in \operatorname{Min} F\}$ and the normal cone of the face $F \cap T$ in $T$ is obtained by an orthogonal projection of this Voronoi domain onto $T$. If different inner products are used, the resulting cones may differ, as seen in the cases of [**144**] and [**192**].

**3.4.5. Finiteness of $G$-perfect forms.** In case of $T = T_G$ being the space of invariant forms of a finite subgroup $G$ of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$, we say that two PQFs $Q$ and $Q'$ are $(G,\boldsymbol{t})$-equivalent ($G$-equivalent, in case $m=1$), if they are $(T_G,\boldsymbol{t})$-equivalent. We show below that the generalized Voronoi algorithm is finite in case of rational $\boldsymbol{t}$. Before, we review some basic definitions and known facts (see [**49**], Section 2).

The pointwise stabilizer of $T_G$ is called the *Bravais group* of $G$ and is denoted by

$$\mathcal{B}(G) = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U^tQU = Q \text{ for all } Q \in T_G\}.$$

Observe that $\mathcal{B}(G)$ can be strictly larger than $G$. The *normalizer* of a subgroup $G$ of $\mathsf{GL}_d(\mathbb{Z})$ is defined by

$$N(G) = \{N \in \mathsf{GL}_d(\mathbb{Z}) : N^{-1}GN = G\}.$$

We generalize these notions, simply by setting $\mathcal{B}_{\boldsymbol{t}}(G) = \mathcal{B}(G) \cap \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ and $N_{\boldsymbol{t}}(G) = N(G) \cap \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$.

The following proposition shows that $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$ is the setwise stabilizer of the subspace $T_G$ of $G$-invariant forms.

PROPOSITION 3.24. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$. Let $G$ be a finite subgroup of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. Then*

$$N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G)) = \{U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) : U^tT_GU = T_G\}.$$

PROOF. We adapt the proof of Lemme 3.2 in [**144**]:

$$\begin{array}{cc}
 & U \in \{U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) : U^tT_GU = T_G\} \\
\Leftrightarrow & \text{for all } Q \in T_G \text{ holds } U^tQU \in T_G \\
\Leftrightarrow & \text{for all } Q \in T_G \text{ and for all } g \in G \ : \ g^tU^tQUg = U^tQU \\
\Leftrightarrow & \text{for all } Q \in T_G \text{ and for all } g \in B_{\boldsymbol{t}}(G) \ : \ g^tU^tQUg = U^tQU \\
\Leftrightarrow & \text{for all } Q \in T_G \text{ and for all } g \in B_{\boldsymbol{t}}(G) \ : \ (UgU^{-1})^tQ(UgU^{-1}) = Q \\
\Leftrightarrow & \text{for all } g \in B_{\boldsymbol{t}}(G) \ : \ (UgU^{-1}) \in B_{\boldsymbol{t}}(G) \\
\Leftrightarrow & U \in N_{\boldsymbol{t}}(B_{\boldsymbol{t}}(G))
\end{array}$$

□

Thus by the foregoing proposition, two PQFs $Q$ and $Q'$ in $T_G$ (i.e. two $G$-perfect forms) are $(G, \boldsymbol{t})$-equivalent, if there exists a $U \in N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$ with $U^t Q U = Q'$. Note that $Q$ and $Q'$ in $T_G$ can be $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent, without being $G$-equivalent. In the classical case it has been shown that up to scaling there are only finitely many $G$-inequivalent, $G$-perfect forms (see [**174**]). Here we give a proof of this fact which also yields the following more general result:

THEOREM 3.25. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$. Let $G$ be a finite subgroup of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. Then up to scaling and $(G, \boldsymbol{t})$-equivalence there exist only finitely many $(G, \boldsymbol{t})$-perfect forms.*

The proof is at the end of this section. It is based on a more general lemma, which allows us not only to generalize the classical theory very easily to the periodic one, but also allows us to obtain a similar finiteness result in Section 4.3.3, in another context. In a way, the proof of Lemma 3.26 below is essentially an adaption and generalization of the proofs by Jaquet-Chiffelle [**144**, Théorème 5.2] and Opgenorth [**192**] for $G$-perfect forms. For the statement of the lemma we need the notion of an *open polyhedral subdivision*, which is a non-intersecting decomposition into polyhedral cones which are open with respect to their affine hull.

LEMMA 3.26. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$. Let $G$ be a finite subgroup of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. Let $\mathcal{P}$ be an open polyhedral subdivision of $\mathcal{S}^d_{>0}$ on which $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ acts by $\Delta \mapsto U^t \Delta U$. Suppose that this action gives only finitely many orbits. Define a polyhedral subdivision of $T_G \cap \mathcal{S}^d_{>0}$ by*

$$\mathcal{P}_G = \{\Delta \cap T_G : \Delta \in \mathcal{P}\}.$$

*Then the normalizer $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$ acts on $\mathcal{P}_G$ and there exist only finitely many orbits with respect to this action.*

PROOF. We start with a definition. For $\Delta \in \mathcal{P}$ we define the automorphism group $\operatorname{Aut}_{\boldsymbol{t}}(\Delta) = \{U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) : U^t \Delta U = \Delta\}$. We consider the set

$$\mathcal{O}_{\Delta,G} = \{U^t \Delta U \cap T_G : U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) \text{ and } U^t \Delta U \cap T_G \neq \emptyset\}.$$

The normalizer $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$ stabilizes $T_G$ setwise, and hence it acts on $\mathcal{O}_{\Delta,G}$. We show that $\mathcal{O}_{\Delta,G}$ is a finite union of $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$-orbits, that is, there are $U_1, \dots, U_m \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with

$$\mathcal{O}_{\Delta,G} = \bigcup_{i=1}^{m} \{N^t(U_i^t \Delta U_i \cap T_G) N : N \in N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))\}.$$

Then the statement of the theorem follows because the set $\{\mathcal{O}_{\Delta,G} : \Delta \in \mathcal{P}\}$ is finite by the assumption made on $\mathcal{P}$.

Let $U_1, U_2$ be in $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ so that $U_i^t \Delta U_i \cap T_G \in \mathcal{O}_{\Delta,G}$ for $i = 1, 2$. The group $\mathcal{B}_{\boldsymbol{t}}(G)$ is a subgroup of $\operatorname{Aut}_{\boldsymbol{t}}(U_1^t \Delta U_1)$ and $\operatorname{Aut}_{\boldsymbol{t}}(U_2^t \Delta U_2)$ which can be seen as follows. By definition the group $\mathcal{B}_{\boldsymbol{t}}(G)$ stabilizes the set $U_i^t \Delta U_i \cap T_G$ pointwise. Since $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ operates on $\mathcal{P}$ and $U_i^t \Delta U_i \cap T_G \neq \emptyset$ every element of $\mathcal{B}_{\boldsymbol{t}}(G)$ has to stabilize all of $U_i^t \Delta U_i$ setwise.

Hence, $U_1 \mathcal{B}_{\boldsymbol{t}}(G) U_1^{-1}$ and $U_2 \mathcal{B}_{\boldsymbol{t}}(G) U_2^{-1}$ are subgroups of $\operatorname{Aut}_{\boldsymbol{t}}(\Delta)$. Assume that $U_1 \mathcal{B}_{\boldsymbol{t}}(G) U_1^{-1} = U_2 \mathcal{B}_{\boldsymbol{t}}(G) U_2^{-1}$. Then $N = U_2^{-1} U_1$ is an element of the normalizer $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$. Furthermore, we have

$$N^t(U_2^t \Delta U_2 \cap T_G) N = N^t U_2^t \Delta U_2 N \cap N^t T_G N = U_1^t \Delta U_1 \cap T_G$$

| $d$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| # $\mathcal{E}$-perfect | 1 | 1 | 2 | 5 | 1628 |
| max. $\delta$ | $0.9069\ldots$ | $0.6168\ldots$ | $0.3729\ldots$ | $0.2536\ldots$ | $0.0360\ldots$ |

TABLE 2. Number of classes and maximum densities of $\mathcal{E}$-perfect forms.

because $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$ stabilizes $T_G$ setwise by Proposition 3.24. Therfore different $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$-orbits in $\mathcal{O}_{\Delta,G}$ induce different subgroups of the form $U\mathcal{B}_{\boldsymbol{t}}(G)U^{-1}$ of $\mathrm{Aut}_{\boldsymbol{t}}(\Delta)$.

For every $\Delta \in \mathcal{P}$ the group $\mathrm{Aut}_{\boldsymbol{t}}(\Delta)$ is finite: Let $\mathcal{R}_1, \ldots, \mathcal{R}_k \in \tilde{\mathcal{S}}^d_{\geq 0}$ be the extreme rays spanning the closed polyhedral cone $\overline{\Delta}$. Choose forms $Q_i \in \mathcal{R}_i$ in the following way: If for a pair $\mathcal{R}_i$ and $\mathcal{R}_j$ there is a $U \in \mathrm{Aut}_{\boldsymbol{t}}(\Delta)$ with $U^t\mathcal{R}_iU = \mathcal{R}_j$ then $U^tQ_iU = Q_j$. We have $\overline{\Delta} = \mathrm{cone}\{Q_1, \ldots, Q_k\}$ and $Q = \sum_{i=1}^k Q_i \in \Delta$ and in particular $Q \in \mathcal{S}^d_{>0}$. So, $\mathrm{Aut}_{\boldsymbol{t}}(Q) = \{U \in \mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z}) : U^tQU = Q\}$ is finite. By construction $\mathrm{Aut}_{\boldsymbol{t}}(Q)$ contains $\mathrm{Aut}_{\boldsymbol{t}}(\Delta)$. So, the group $\mathrm{Aut}_{\boldsymbol{t}}(\Delta)$ has only a finite number of different subgroups. Hence, $\mathcal{O}_{\Delta,G}$ is a finite union of $N_{\boldsymbol{t}}(\mathcal{B}_{\boldsymbol{t}}(G))$-orbits. $\square$

The proof of Theorem 3.25 is an immediate consequence.

PROOF OF THEOREM 3.25. Consider the Ryshkov polyhedron $\mathcal{P}_{\lambda,\boldsymbol{t}}$ for some $\lambda > 0$. For each face $F$ of $\mathcal{P}_{\lambda,\boldsymbol{t}}$ we define the (relative) open polyhedral cone

$$\Delta_F = \{\alpha Q : \alpha > 0 \text{ and } Q \in \mathrm{relint}\, F\}.$$

Then by Theorem 3.23, $\mathcal{P} = \{\Delta_F : F \text{ face of } \mathcal{P}_{\lambda,\boldsymbol{t}}\}$ gives an open polyhedral subdivision of $\mathcal{S}^d_{>0}$ as required in Lemma 3.26. $\square$

Let us finally note that Theorem 3.25 gives the possibility for systematic, computational searches of dense periodic sphere packings in higher dimensional spaces. We are convinced that these will lead to new discoveries.

**3.4.6. Eisenstein, Gaussian and Hurwitz quaternionic perfect forms.** As examples for the $G$-theory described above, we consider three cases that have been studied extensively before.

*Eisenstein forms.* If $d$ is even, then $Q \in \mathcal{S}^d_{>0}$ is said to be an *Eisenstein form* if it is invariant with respect to a group $G \subset \mathsf{GL}_d(\mathbb{Z})$ of order 3 acting fixed-point-free on $\mathbb{Z}^d \setminus \{\mathbf{0}\}$ by $z \mapsto Uz$. For example

$$G = \left\langle \mathrm{id}_{d/2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle,$$

where $\otimes$ denotes the *Kronecker product*. A corresponding lattice $L \subset \mathbb{R}^d$ can be viewed as a *complex lattice* of dimension $d/2$ over the *Eisenstein integers*

$$\mathcal{E} = \left\{ a + be^{2\pi i/3} : a, b \in \mathbb{Z} \right\},$$

that is, $L = B\mathcal{E}^{d/2} \subset \mathbb{C}^{d/2}$ with a suitable $B \in \mathsf{GL}_{d/2}(\mathbb{C})$. On the other hand, it can be seen that each complex lattice of this form yields an Eisenstein form.

It turns out that the space of $G$-invariant forms $T_G$ has dimension $(d/2)^2$. Thus for $d = 2$ we find only one Eisenstein form up to scaling, associated to the hexagonal lattice $\mathsf{A}_2$. It is therefore trivially $\mathcal{E}$-perfect (*Eisenstein perfect*). From dimension 4

| $d$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| # $\mathcal{G}$-perfect | 1 | 1 | 1 | 2 | $\geq 17757$ |
| max. $\delta$ | 0.7853... | 0.6168... | 0.3229... | 0.2536... | |

TABLE 3. Number of classes and maximum densities of $\mathcal{G}$-perfect forms.

on the situation is already more interesting. In Table 2 we list number of classes and maximum sphere packing densities of $\mathcal{E}$-perfect forms up to dimension 10.

For $d = 4$, the Ryshkov polyhedron is 4-dimensional in $\mathcal{S}^d$. Up to $\mathcal{E}$-equivalence (by mappings $Q \mapsto U^t QU$ preserving $T_G$), there is only one $\mathcal{E}$-perfect form, namely the one associated to the lattice $\mathsf{D}_4$. Consequently the Voronoi graph (up to $\mathcal{E}$-equivalence) is just a single vertex with a loop. In dimension 6, we find already two $\mathcal{E}$-inequivalence $\mathcal{E}$-perfect forms, associated to the lattices $\mathsf{E}_6$ and its dual $\mathsf{E}_6^*$.

The classification of Eisenstein forms in dimension 8 was almost finished by Sigrist in [**235**]. He found all five classes of $\mathcal{E}$-perfect forms and their neighboring relations. However, he could not rule out the existence of other *$\mathcal{E}$-contiguous* neighbors of the forms associated to $\mathsf{E}_8$. Recently we finished the classification in dimension 8, and moreover found that the number of $\mathcal{E}$-perfect classes "explodes" to 1628 in dimension 10 (see [**224**] for details).

Note that the largest known lattice sphere packing density $\delta$ is attained among $\mathcal{E}$-perfect forms up to dimension 10. A noteworthy phenomenon that occurs among $\mathcal{E}$-perfect forms in dimension 10 is the existence of $\mathcal{E}$-inequivalent $\mathcal{E}$-perfect forms which are nevertheless arithmetically equivalent. This happens for two arithmetically equivalent forms associated to the lattice $K_{10}'$ (see [**174**, Section 8.5]).

*Gaussian forms.* For even $d$, a *Gaussian form* $Q \in \mathcal{S}^d_{>0}$ is defined as a form containing a group $G \subset \mathsf{GL}_d(\mathbb{Z})$ of order 4 in their automorphism group, acting fixed-point-free on $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. For example

$$G = \left\langle \mathrm{id}_{d/2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle .$$

A corresponding lattice $L \subset \mathbb{R}^d$ can be viewed as a complex lattice of dimension $d/2$ over the *Gaussian integers*

$$\mathcal{G} = \{a + bi : a, b \in \mathbb{Z}\} .$$

Vice versa, every such lattice yields a Gaussian form.

As in the case of Eisenstein forms, it turns out that $T_G$ has dimension $(d/2)^2$. For $d = 2$ we find only one $\mathcal{G}$-perfect (*Gaussian perfect*) form up to scaling, namely $\mathbb{Z}^2$. The only $\mathcal{G}$-perfect forms in dimension 6 and 8 are associated to the lattices $\mathsf{D}_6$, $\mathsf{D}_8$ and $\mathsf{E}_8$. As shown in Table 3, the number of classes of $\mathcal{G}$-perfect forms in dimension 10 grows even beyond the corresponding number of classes of $\mathcal{E}$-perfect forms. So far we were not able to finish the classification, but we think it is computationally within reach on a suitable machine. As in the case of $\mathcal{E}$-perfect forms, the enumeration in dimension 8 was started by Sigrist [**235**]. However, he did not finish the classification of *$\mathcal{G}$-contiguous* $\mathcal{G}$-perfect neighbors of $\mathsf{E}_8$. Our computations shows that his list was nevertheless complete.

*Hurwitz quaternionic forms.* For $d$ divisible by 4, a form $Q \in \mathcal{S}^d_{>0}$ is called *Hurwitz quaternionic* if it is invariant with respect to a group $G \subset \mathsf{GL}_d(\mathbb{Z})$ isomorphic to $2A_4$ and acting fixed-point-free on $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. Here, $A_4$ denotes the *alternating*

| $d$ | 4 | 8 | 12 | 16 |
|---|---|---|---|---|
| # $\mathcal{H}$-perfect | 1 | 1 | 8 | ? |
| max. $\delta$ | $0.6168\ldots$ | $0.2536\ldots$ | $0.03125\ldots$ | $0.01471\ldots$ |

Table 4. Number of classes and maximum densities of $\mathcal{H}$-perfect forms.

*group* of degree 4. There is a correspondence between Hurwitz quaternionic forms and lattices in $\mathbb{R}^d$ which can be viewed as *Hurwitz quaternionic lattices* over the *Hurwitz quaternionic integers*

$$\mathcal{H} = \left\{ a + bi + cj + dk : a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \tfrac{1}{2} \right\} .$$

We refer to [**69**, Section 2.6] for details.

It turns out that $T_G$ is of dimension $\binom{d/2}{2}$. This leaves only one Hurwitz quaternionic form and therefore only one *$\mathcal{H}$-perfect form* up to scaling for $d = 4$, which is associated to $\mathsf{D}_4$. There is also only one class of $\mathcal{H}$-perfect forms in dimension 8, corresponding to $\mathsf{E}_8$.

The situation becomes more interesting in dimension 12. There are precisely eight classes of $\mathcal{H}$-perfect forms, as previously observed by Jaquet-Chiffelle and Sigrist (cf. [**236**]). All of them are also perfect in the classical sense.

A quite interesting consequence of the classification in dimension 12 is the possibility to derive a sharp bound for the largest possible sphere packing density among Hurwitz quaternionic forms in dimension 16. Using a *Mordell type inequality*, Vance [**257**] shows that the *Barnes-Wall lattice* $\mathsf{BW}_{16}$ has the largest density among lattices with a Hurwitz quaternionic structure in dimension 16. A very nice example of a human-computer-interacted proof!

CHAPTER 4

# Voronoi II

In this chapter we generalize *Voronoi's second reduction theory*, which is based on Delone subdivisions. We give the necessary background in Section 4.1. Our extension of Voronoi's theory is twofold, and similar to the extensions discussed in Chapter 3. On the one hand we extend to periodic sets (Section 4.2), on the other hand we develop a $T$-theory for linear subspaces $T \subseteq \mathcal{S}^d$ (Section 4.3). The latter extension is of particular importance for the applications in Chapter 5, as it allows for example a systematic enumeration of all Delone subdivisions with prescribed properties (for example those with a given finite group of symmetries). In all, we do keep an eye on practical computability. Finally, in Section 4.4, we discuss Delone polytopes and complexes in further detail, with particular emphasis on Delone simplices.

## 4.1. Voronoi's second reduction

**4.1.1. Delone polyhedra and Delone subdivisions.** Given a discrete set $\Lambda' \subset \mathbb{R}^d$, a polytope $P = \operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ with vertices $\boldsymbol{v}_i \in \Lambda'$ is called a *Delone polytope*, if there exists a center $\boldsymbol{c} \in \mathbb{R}^d$ and a radius $r > 0$ such that the Euclidean distance between $\boldsymbol{c}$ and points $\boldsymbol{v} \in \Lambda'$ satisfies $\|\boldsymbol{c} - \boldsymbol{v}\| \geq r$, with equality only for the vertices of $P$. The set of all Delone polytopes of a lattice forms a polyhedral subdivision of $\mathbb{R}^d$.

Our main interest is in discrete *vertex-sets* which are periodic, say

$$\Lambda' = \bigcup_{i=1}^{m} \boldsymbol{t}_i' + L$$

with a lattice $L$ and $\boldsymbol{t}_i' \in \mathbb{R}^d$ for $i = 1, \dots, m$. As in Section 3.2, it is in many cases convenient to work with coordinates with respect to a given basis $A \in \mathsf{GL}_d(\mathbb{R})$ of $L$. This means that we work with the standard periodic sets

$$\Lambda = \Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d \tag{4.1}$$

(with $\Lambda' = A\Lambda$) and norm defined by $Q_A = A^t A$. Hence the norm of $\boldsymbol{x} \in \mathbb{R}^d$ is given by $\sqrt{Q_A[\boldsymbol{x}]}$. A polytope $P = \operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ with $\boldsymbol{v}_i \in \Lambda$, is called a *Delone polytope* of $Q_A$ and $\Lambda$, if there exists a $\boldsymbol{c} \in \mathbb{R}^d$ and a real number $r$ with

$$Q_A[\boldsymbol{c} - \boldsymbol{v}] \geq r^2,$$

where equality holds if and only if $\boldsymbol{v}$ is a vertex of $P$.

It is a bit more general (and in some situations convenient) to consider positive semidefinite forms $Q$, as proposed by Namikawa in [**186**]. Positive semidefinite forms define seminorms on $\mathbb{R}^d$ by $\sqrt{Q[\boldsymbol{x}]}$. A (possibly unbounded) polyhedron

$P = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ with $\boldsymbol{v}_i \in \Lambda$ is then called a *Delone polyhedron* of $Q$ and $\Lambda$, if there exists a $\boldsymbol{c} \in \mathbb{R}^d$ and a real number $r$ with $Q[\boldsymbol{c} - \boldsymbol{v}_i] = r^2$ for $i = 1, 2, \ldots$ and $Q[\boldsymbol{c} - \boldsymbol{v}] > r^2$ for all $\boldsymbol{v} \in \Lambda \setminus \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$.

Suppose for $Q \in \mathcal{S}^d_{\geq 0}$ there exists a matrix $U \in \mathsf{GL}_d(\mathbb{Z})$ and a $Q' \in \mathcal{S}^{d'}_{>0}$ with

$$U^t Q U = \begin{pmatrix} 0 & 0 \\ 0 & Q' \end{pmatrix} = \bar{Q}. \tag{4.2}$$

Then $d' = \operatorname{rank} Q$ and if $P$ is a Delone polyhedron of $Q$ then $U^{-1}P$ is a Delone polyhedron of $\bar{Q}$. Latter are of the form $\mathbb{R}^{d-d'} \times P'$ with $P' \in \mathbb{R}^{d'}$ a Delone polytope of $Q'$ (see [**186**]). It turns out that the set of all forms being arithmetical equivalent to some positive semidefinite form $\bar{Q}$ of the form (4.2) has a nice description in terms of the rational closure $\tilde{\mathcal{S}}^d_{\geq 0}$ of $\mathcal{S}^d_{>0}$ (cf. (3.17)).

Proposition 4.1. *The set of all forms being arithmetical equivalent to some positive semidefinite form $\bar{Q}$ of the form* (4.2)*, with positive definite $Q'$, is equal to the rational closure $\tilde{\mathcal{S}}^d_{\geq 0}$.*

Proof. Suppose $Q$ is arithmetical equivalent to some positive semidefinite form $\bar{Q}$ of the form (4.2), with $Q'$ positive definite. Then $Q'$ is a positive sum of rank-1 forms (cf. [**262**, Section 24]). Therefore $\bar{Q}$ and $Q$ are of this form as well.

Suppose on the other hand, $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$. Then, by definition of $\tilde{\mathcal{S}}^d_{\geq 0}$, $Q$ is the positive sum of rank 1-forms, e.g. $Q = \sum_{i=1}^m \alpha_i \boldsymbol{v}_i \boldsymbol{v}_i^t$ with $\boldsymbol{v}_i \in \mathbb{Z}^d$ and $\alpha_i > 0$ for $i = 1, \ldots, m$. If $\operatorname{rank} Q = d$ there is nothing to show. If $\operatorname{rank} Q < d$, then there exist linearly independent $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k \in \mathbb{R}^d$ with $k = d - \operatorname{rank} Q$ such that $Q[\boldsymbol{p}_j] = \sum_{i=1}^m \alpha_i (\boldsymbol{p}_j^t \boldsymbol{v}_i)^2 = 0$ for $j = 1, \ldots, k$. Thus the $\boldsymbol{p}_j$ are orthogonal to each of the $\boldsymbol{v}_i$ and therefore they span a $k$-dimensional linear subspace which contains a $k$-dimensional sublattice of $\mathbb{Z}^d$. We choose a basis $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ of this sublattice and extend it to a basis $U = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$ of $\mathbb{Z}^d$. Then $(U^{-1})^t Q (U^{-1})$ is of the desired form (4.2). ☐

The importance of the rational closure is due to the following observation.

Proposition 4.2. *For $Q \in \mathcal{S}^d_{\geq 0}$ exists a $d$-dimensional Delone polyhedron with respect to a standard periodic vertex-set if and only if $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$.*

Although this proposition might be known we are not aware of a reference. In [**186**, §2.1] Namikawa showed that every $Q$ lying in the rational closure has a $d$-dimensional Delone polyhedron. For completeness we repeat his argument.

Proof. If $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$, there exist $d$-dimensional Delone polyhedra which after a suitable transformation in $\mathsf{GL}_d(\mathbb{Z})$ are those of the form $\bar{Q}$ in (4.2). If on the other hand $Q \notin \tilde{\mathcal{S}}^d_{\geq 0}$, then for all arithmetical equivalent forms $\bar{Q}$ as in (4.2), the form $Q' \in \mathcal{S}^{d'}_{\geq 0}$ is not positive definite, hence $d' > \operatorname{rank}(Q)$. For an arithmetical equivalent form $\bar{Q}$ with minimal $d'$ there exists no rational, hence no integral vector in the kernel of $Q'$. As a consequence, for such a $Q'$ we find for all $\boldsymbol{c} \in \mathbb{R}^{d'}$ and all $r > 0$ a $\boldsymbol{v} \in \mathbb{Z}^{d'}$ with $Q'[\boldsymbol{v} - \boldsymbol{c}] < r$ (see [**234**, Lecture VI]). Therefore there do not exist $d'$-dimensional Delone polyhedra for $Q'$ and $\mathbb{Z}^{d'}$, and hence no $d$-dimensional ones for $Q$ and $\mathbb{Z}^d$ as well. The same is true for $Q$ and a standard periodic vertex-set $\Lambda$. ☐

The set $\mathrm{Del}_\Lambda(Q)$ of all Delone polyhedra of a $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$ with vertices in the standard periodic vertex-set $\Lambda$ is called the *Delone subdivision* of $Q$ (with respect to $\Lambda$). If all elements in $\mathrm{Del}_\Lambda(Q)$ are simplices, $\mathrm{Del}_\Lambda(Q)$ is called a *triangulation.* The subdivision $\mathrm{Del}_\Lambda(Q)$ is a polyhedral subdivision of $\mathbb{R}^d$ which is invariant under translations of the form $\boldsymbol{x} \mapsto \boldsymbol{x}+\boldsymbol{v}$, where $\boldsymbol{v} \in \mathbb{Z}^d$. Therefore $\mathrm{Del}_\Lambda(Q)$ is completely determined by the *stars* of the translation vectors (vertices) $\boldsymbol{t}'_i$ in (4.1), for $i = 1, \dots, m$. Here, a star of a single vertex is the set of all Delone polyhedra containing it.

We call two Delone polyhedra $P$ and $P'$ *(translational) equivalent* if there exists a $\boldsymbol{v} \in \mathbb{R}^d$ so that $P = \boldsymbol{v} + P'$. Two Delone polyhedra $P$ and $P'$ with integral vertices are called $\mathsf{GL}_d(\mathbb{Z})$*-equivalent* if there exists a $\boldsymbol{v} \in \mathbb{Z}^d$ and $U \in \mathsf{GL}_d(\mathbb{Z})$ such that $P = \boldsymbol{v} + UP'$. We also consider equivalence of Delone polyhedra $P$ and $P'$ with vertices in a standard periodic vertex-set $\Lambda$, where $\Lambda$ in (4.1) is defined by some $\boldsymbol{t} = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_{m-1}) \in \mathbb{R}^{d\times(m-1)}$ and by setting $\boldsymbol{t}_m = \mathbf{0}$. Recall that the automorphism group of $\Lambda$ is denoted by $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$. Therefore, $P$ and $P'$ are called $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$*-equivalent*, if there exists a $U \in \mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$ and a $\boldsymbol{v} \in \Lambda_{\boldsymbol{t}} - \Lambda_{\boldsymbol{t}} = \{\boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x}, \boldsymbol{y} \in \Lambda_{\boldsymbol{t}}\}$ with $P = \boldsymbol{v} + UP'$.

We say that $\mathrm{Del}_\Lambda(Q)$ is a *refinement* of $\mathrm{Del}_\Lambda(Q')$ (and $\mathrm{Del}_\Lambda(Q')$ is a *coarsening* of $\mathrm{Del}_\Lambda(Q)$), if every Delone polytope of $Q$ is contained in a Delone polytope of $Q'$.

**4.1.2. Voronoi's theory of secondary cones.** Before we generalize the (second) reduction theory of Voronoi in the next section, we briefly recall the original theory (see [**263**], [**76**]). Recall from Section 2.1 that the general task of reduction in the classical theory of PQFs is to find a fundamental domain in $\mathcal{S}^d_{>0}$ with respect to the action of $\mathsf{GL}_d(\mathbb{Z})$. Voronoi's second reduction theory is based on *secondary cones*

$$\boldsymbol{\Delta}(\mathcal{D}) = \{Q \in \tilde{\mathcal{S}}^d_{\geq 0} : \mathrm{Del}_{\mathbb{Z}^d}(Q) = \mathcal{D}\}$$

of Delone triangulations $\mathcal{D}$ with vertex-set $\mathbb{Z}^d$. In the literature these are often referred to as *L-type domains.* We prefer the term "secondary cone" because of the close connection to the theory of *secondary polytopes* (see [**74**]). Note that there is no problem in extending the definition of a secondary cone $\boldsymbol{\Delta}(\mathcal{D})$ to Delone subdivisions $\mathcal{D}$ of a fixed standard periodic vertex-set $\Lambda$.

For our formulation of Voronoi's theory and its generalizations described later, we define for an affinely independent set $V \subseteq \mathbb{R}^d$ of cardinality $d+1$ and a point $\boldsymbol{w} \in \mathbb{R}^d$ the quadratic form

$$N_{V,\boldsymbol{w}} = \boldsymbol{w}\boldsymbol{w}^t - \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \boldsymbol{v}\boldsymbol{v}^t, \tag{4.3}$$

where the coefficients $\alpha_{\boldsymbol{v}}$ are uniquely determined by the affine dependency

$$\boldsymbol{w} = \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \boldsymbol{v} \quad \text{with} \quad 1 = \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}}.$$

In the special situation of $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}\}$ being the vertices of a Delone simplex $L$ and $\boldsymbol{w}$ being the additional vertex of a Delone simplex $L' = \mathrm{conv}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_{d+1}, \boldsymbol{w}\}$ *adjacent* to $L$, we write $N_{L,L'}$ for $N_{V,\boldsymbol{w}}$. To describe secondary cones of Delone triangulations, Voronoi uses so-called *regulators*, denoted by $\varrho_{(L,L')}$, which are positive multiples of the linear forms $\langle N_{L,L'}, \cdot\rangle$ on $\mathcal{S}^d$. Using our terminology his description can be given as follows.

THEOREM 4.3. *([**263**, §77]) Let $Q$ be a PQF whose Delone subdivision* $\mathrm{Del}_{\mathbb{Z}^d}(Q)$ *is a triangulation. Then the secondary cone of* $\mathrm{Del}_{\mathbb{Z}^d}(Q)$ *is the full-dimensional open polyhedral cone*

$$\mathbf{\Delta}(\mathrm{Del}_{\mathbb{Z}^d}(Q)) = \{Q' \in \mathcal{S}^d : \langle N_{L,L'}, Q' \rangle > 0 \textit{ for all adjacent } L, L' \in \mathrm{Del}_{\mathbb{Z}^d}(Q)\}.$$

EXAMPLE 4.4. As an example, we describe the Delone subdivision of *Voronoi's principal form of the first type*

$$Q[\boldsymbol{x}] = d\sum_{i=1}^{d} x_i^2 - \sum_{i=1}^{d} x_i x_j$$

in greater detail. Note that this PQF is associated to the lattice $\mathsf{A}_d^*$, the *dual* of the root lattice $\mathsf{A}_d$. The Delone subdivision of $Q$ is a triangulation and can be described as follows: Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the standard basis vectors of $\mathbb{Z}^d$, and set $\boldsymbol{e}_{d+1} = -\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_d$. For a permutation $\pi \in \mathrm{Sym}(d+1)$ we define the $d$-dimensional simplex $L_\pi$ by

$$L_\pi = \mathrm{conv}\{\boldsymbol{e}_{\pi(1)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\}.$$

The set of simplices $\{L_\pi + \boldsymbol{v} : \boldsymbol{v} \in \mathbb{Z}^d, \pi \in \mathrm{Sym}(d+1)\}$ defines a triangulation $\mathcal{D}_1$ of $\mathbb{R}^d$. The full-dimensional cells containing the origin are $L_\pi$, $\pi \in \mathrm{Sym}(d+1)$. Two simplices $L_\pi$ and $L_{\pi'}$ have a facet in common if and only if $\pi$ and $\pi'$ differ by a single transposition of two adjacent positions. The automorphism group of $\mathcal{D}_1$ is isomorphic to the permutation group $\mathrm{Sym}(d+1)$. This consists of all Delone polytopes containing the origin.

From this one derives by Theorem 4.3 that the secondary cone of $\mathcal{D}_1$ is (cf. [**263**, §102–104])

$$\mathbf{\Delta}(\mathcal{D}_1) = \{Q \in \mathcal{S}^d : q_{ij} < 0 \text{ for } i \neq j \text{ and } \textstyle\sum_{i=1}^d q_{ij} > 0 \text{ for } j = 1, \ldots, d\ \}.$$

Its topological closure $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$ is called *Voronoi's principal domain of the first type.*

We say that two secondary cones of Delone triangulations are *bistellar neighbors* if the Delone triangulations differ by a *bistellar flip*, which is a specific change of the triangulation. We explain those and a generalization in Section 4.3.2. With this terminology at hand, we can state the main part of Voronoi's second reduction theory in the following way.

THEOREM 4.5 (*Voronoi's Second Reduction Theory).*

I. *The topological closures* $\overline{\mathbf{\Delta}(\mathcal{D})}$ *give a polyhedral subdivision of* $\tilde{\mathcal{S}}^d_{\geq 0}$. *The closures of two secondary cones have a common facet if and only if they are bistellar neighbors.*
II. *The group* $\mathsf{GL}_d(\mathbb{Z})$ *acts on the subdivision by* $\overline{\mathbf{\Delta}(\mathcal{D})} \mapsto U^t\overline{\mathbf{\Delta}(\mathcal{D})}U$. *Under this group action there are only finitely many inequivalent secondary cones.*

Note that the polyhedral subdivision of $\mathcal{S}^d_{>0}$ (as well as of $\tilde{\mathcal{S}}^d_{\geq 0}$) into secondary cones is similar to the subdivision of $\mathcal{S}^d_{>0}$ into Voronoi domains (see Section 3.1.8). It is also an open polyhedral subdivision, that is, every cone of the subdivision of $\tilde{\mathcal{S}}^d_{\geq 0}$ is an open polyhedral cone with respect to its affine hull. In fact, in dimension 2 and 3 the subdivision into Voronoi domains and the subdivision into secondary cones even coincide. In dimension 4 and 5 the subdivision into secondary cones is a true refinement of the other one. That is, each secondary cone is contained in a

| Dimension | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| # Delone triangulations | 1 | 1 | 1 | 3 | 222 |

Table 1. Numbers of inequivalent Delone triangulations.

Voronoi domain. From dimension 6 on however, such a relation no longer holds (cf. [**106**]).

Finally let us note, as in the case of Voronoi's first reduction, we can subdivide the secondary cones into smaller cones and obtain a reduction domain in the strict sense introduced in Section 2.1.

**4.1.3. Enumerating Delone subdivisions.** Theorem 4.5 translates into an algorithm to enumerate all inequivalent Delone triangulations in a given dimension. We developed the program `scc` (*secondary cone cruiser*) which is an implementation of Algorithm 3. The interested reader can download `scc` [**286**]. Using our implementation it is possible to reproduce the classification of all inequivalent Delone triangulations up to dimension 5. Table 1 shows the numbers.

**Input:** Dimension $d$.
**Output:** Set $\mathcal{R}$ of all inequivalent $d$-dimensional Delone triangulations.
$Y \leftarrow \{\mathcal{D}_1\}$, where $\mathcal{D}_1$ is the Delone triangulation described in Example 4.4.
$\mathcal{R} \leftarrow \emptyset$.
**while** there is a $\mathcal{D} \in Y$ **do**
  $Y \leftarrow Y \backslash \{\mathcal{D}\}$. $\mathcal{R} \leftarrow \mathcal{R} \cup \{\mathcal{D}\}$.
  compute the regulators of $\mathcal{D}$.
  compute the facets $F_1, \ldots, F_n$ of $\overline{\mathbf{\Delta}(\mathcal{D})}$.
  **for** $i = 1, \ldots, n$ **do**
    compute the bistellar neighbor $\mathcal{D}_i$ of $\mathcal{D}$ which is defined by $F_i$.
    **if** $\mathcal{D}_i$ is not equivalent to a Delone triang. in $\mathcal{R} \cup \{\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}\}$ **then**
      $Y \leftarrow Y \cup \{\mathcal{D}_i\}$.
    **end if**
  **end for**
**end while**

Algorithm 3. Enumeration of all inequivalent Delone triangulations.

We are not the first to compute this classification. Voronoi performed the classification of all inequivalent Delone triangulations up to dimension 4 in his memoir [**263**]. In [**21**] and [**216**] Ryshkov and Baranovskii reported on 221 inequivalent Delone triangulation in dimension 5. However, they missed one type which was found by Engel [**101**]. Grishukhin and Engel [**103**] undertook the non-trivial task of identifying the Delone triangulation missing in the list of Ryshkov and Baranovskii. There they also report on several errors in both lists. Beginning with dimension 6 the number of inequivalent Delone triangulations starts to explode. We do not know how many inequivalent triangulations we have to expect in dimension 6. An example for the difficulties in dimension 6 is seen in the proof of Theorem 5.28. There we consider the $3^{40}$ refining triangulations for the Delone subdivision of the lattice $\mathsf{E}_6^*$ (the dual of the root lattice $\mathsf{E}_6$).

| Dimension | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| # Delone subdivisions | 1 | 2 | 5 | 52 | 179,372 |

TABLE 2. Numbers of inequivalent Delone subdivisions.

Arbitrary Delone subdivisions are limiting cases of Delone triangulations. Their secondary cones occur on the boundaries of full-dimensional secondary cones of Delone triangulations. The following proposition seems to be folklore. One can find a proof for example in [**254**, Prop. 2.6.1]:

PROPOSITION 4.6. *Let $\mathcal{D}$ be a Delone triangulation.*

(1) *A positive semidefinite quadratic form $Q$ lies in $\overline{\mathbf{\Delta}(\mathcal{D})}$ if and only if $\mathcal{D}$ is a refinement of* $\mathrm{Del}_{\mathbb{Z}^d}(Q)$.
(2) *If two positive semidefinite quadratic forms $Q$ and $Q'$ both lie in $\overline{\mathbf{\Delta}(\mathcal{D})}$, then* $\mathrm{Del}_{\mathbb{Z}^d}(Q + Q')$ *is a common refinement of* $\mathrm{Del}_{\mathbb{Z}^d}(Q)$ *and* $\mathrm{Del}_{\mathbb{Z}^d}(Q')$.

By this proposition, the classification of all inequivalent Delone subdivisions is equivalent to the classification of all inequivalent secondary cones. This has been done up to dimension 5. The 1- and 2-dimensional cases are trivial; the 3-dimensional case goes back to Federov, who classified all polytopes which tile 3-dimensional space by translates in 1885. Delone [**75**] (later corrected by Stogrin [**245**]) found 51 of the 52 Delone subdivisions in dimension 4. Engel [**102**] reports that there are $179,372$ inequivalent five-dimensional Delone subdivisions. It is in principle possible to verify his result by enumerating the faces of the 222 secondary cones of Delone triangulations in dimension 5 up to $\mathsf{GL}_d(\mathbb{Z})$-equivalence (using for example methods as described in Appendix A). Table 2 summarizes the discussion.

## 4.2. Extension to periodic sets

**4.2.1. Polyhedral description of secondary cones.** Suppose $\mathcal{D}$ is a Delone subdivision with respect to a standard periodic vertex-set $\Lambda$. In this section we want to describe the *secondary cone*

$$\mathbf{\Delta}(\mathcal{D}) = \{Q \in \tilde{\mathcal{S}}^d_{\geq 0} : \mathrm{Del}_\Lambda(Q) = \mathcal{D}\}$$

explicitly, by linear equalities and inequalities. The description is needed for our applications in Chapter 5 and therefore we put some effort into avoiding redundancies.

Theorem 4.7 below gives the precise statement. It generalizes Voronoi's "fundamental theorem" for Delone triangulations to arbitrary polyhedral subdivisions. Moreover, we allow standard periodic vertex-sets and degenerate (unbounded) polyhedra. Our proof in Section 4.2.2 not only generalizes Voronoi's theory, but also shortens and simplifies his argumentation. The linear equalities are coming from $d$-dimensional non-simplex polyhedra in $\mathcal{D}$. Hence, in the generic case of Delone triangulations there are no linear equalities. The linear inequalities are coming from $(d-1)$-dimensional polyhedra (facets) in $\mathcal{D}$. For the formulation of these linear conditions, recall the definition of $N_{V,\boldsymbol{w}}$ in (4.3). The *vertex-set* of a polyhedron $P$, denoted by $\operatorname{vert} P$, is defined as the set $\Lambda \cap P$.

THEOREM 4.7. I. *Let $\mathcal{D}$ be a polyhedral subdivision of $\mathbb{R}^d$ with a standard periodic vertex-set $\Lambda$. Then the closure $\overline{\mathbf{\Delta}(\mathcal{D})}$ is a polyhedral cone in $\tilde{\mathcal{S}}^d_{\geq 0}$ and $\mathbf{\Delta}(\mathcal{D})$ is the set of all $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$ satisfying*

(a) *for every $d$-dimensional polyhedron $P \in \mathcal{D}$ the equalities*

$$\langle N_{V,\boldsymbol{w}}, Q\rangle = 0,$$

*for one (which can be chosen arbitrarily) affinely independent set of $d+1$ vertices $V \subseteq \operatorname{vert} P$ and all $\boldsymbol{w} \in \operatorname{vert} P$;*

(b) *for every $(d-1)$-dimensional polyhedron $F \in \mathcal{D}$ the inequality*

$$\langle N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}, Q\rangle > 0,$$

*for one (which can be chosen arbitrarily) affinely independent set of $d$ vertices $V \subseteq \operatorname{vert} F$ and two vertices $\boldsymbol{w} \in \operatorname{vert} P \setminus F$ and $\boldsymbol{w}' \in \operatorname{vert} P' \setminus F$ of the two adjacent $d$-dimensional polyhedra $P, P' \in \mathcal{D}$ with $F = P \cap P'$.*

II. *The map $\mathcal{D} \mapsto \overline{\boldsymbol{\Delta}(\mathcal{D})}$ gives an isomorphism between the poset of Delone subdivisions of $\Lambda$ ordered by coarsening and the poset of closures of secondary cones ordered by inclusion. The closures of all secondary cones of Delone subdivisions form a polyhedral subdivision of $\tilde{\mathcal{S}}^d_{\geq 0}$.*

Note that different choices of $V$, $\boldsymbol{w}$ and $\boldsymbol{w}'$ for the inequalities could yield different conditions, respectively different forms $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}$. Nevertheless, by our theorem, these are the same on the linear subspace $U$ defined by the equalities. In other words, their orthogonal projections $\pi_U(N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'})$ onto $U$ are all positive multiples of a uniquely determined form $N_{\mathcal{D},F} \in U$ with $\langle N_{\mathcal{D},F}, N_{\mathcal{D},F}\rangle = 1$.

Note also that for every $\boldsymbol{v} \in \mathbb{R}^d$ we have

$$N_{V+\boldsymbol{v},\boldsymbol{w}+\boldsymbol{v}} = N_{V,\boldsymbol{w}}. \tag{4.4}$$

Therefore, and because the vertex-set of the subdivision is assumed to be periodic, the theorem gives only finitely many inequalities. This shows that $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ is a polyhedral cone.

Let us also remark that the theorem is valid for arbitrary periodic sets, that is, finite unions of lattice translates $\boldsymbol{t}'_i + L$, if we replace $\tilde{\mathcal{S}}^d_{\geq 0}$ by $(A^{-1})^t\tilde{\mathcal{S}}^d_{\geq 0}(A^{-1})$ where $A \in \mathsf{GL}_d(\mathbb{R})$ defines the lattice $L = A\mathbb{Z}^d$.

Finally, note that in the terminology introduced in Section 3.2, Theorem 4.7 deals only with periodic sets represented by $X = (Q, \boldsymbol{t}) \in \mathcal{S}^{d,m}_{>0}$, where the translational part $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$ is fixed. So we have only developed a "$\boldsymbol{t}$-theory" analogous to the one in Section 3.4.2.

**4.2.2. Proof of the fundamental theorem.** In this section we prove Theorem 4.7. We first give two propositions, both dealing with the redundancies in the set of equations and inequalities, which one would obtain by considering all possible choices of $V$, $\boldsymbol{w}$ and $\boldsymbol{w}'$. Proposition 4.8 takes care of the equalities and Proposition 4.10 of the inequalities. The latter shows that the orthogonal projections of all the forms $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}$ for a facet $F$, onto the linear subspace defined by the equalities, are unique up to positive multiples.

Proposition 4.8. *Let $Q \in \mathcal{S}^d$ and $V \subset \mathbb{R}^d$ be an affinely independent set of cardinality $d+1$. Let $\boldsymbol{c} \in \mathbb{R}^d$ and $r > 0$ be such that $Q[\boldsymbol{c}-\boldsymbol{v}] = r^2$ for all $\boldsymbol{v} \in V$. Then*

$$Q[\boldsymbol{w}-\boldsymbol{c}] - r^2 = \langle Q, N_{V,\boldsymbol{w}}\rangle.$$

Proof. The proof is straightforward. We have

$$\begin{aligned} Q[\boldsymbol{w}-\boldsymbol{c}]-r^2 &= \langle Q, \boldsymbol{w}\boldsymbol{w}^t\rangle + \langle Q, -2\boldsymbol{w}\boldsymbol{c}^t + \boldsymbol{c}\boldsymbol{c}^t\rangle - r^2 \\ &= \langle Q, \boldsymbol{w}\boldsymbol{w}^t\rangle + \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \langle Q, -2\boldsymbol{v}\boldsymbol{c}^t + \boldsymbol{c}\boldsymbol{c}^t\rangle - r^2, \end{aligned}$$

with $\alpha_{\boldsymbol{v}}$ as in (4.3). For each $\boldsymbol{v} \in V$ we use the equality $Q[\boldsymbol{v}-\boldsymbol{c}] = r^2$ which is equivalent to $-\langle Q, \boldsymbol{v}\boldsymbol{v}^t\rangle = \langle Q, -2\boldsymbol{v}\boldsymbol{c}^t + \boldsymbol{c}\boldsymbol{c}^t\rangle - r^2$. This yields the desired expression. □

Remark 4.9. By Proposition 4.8 the sign of $\langle Q, N_{V,\boldsymbol{w}}\rangle$ has the following interpretation: If it is positive, then $\boldsymbol{w}$ lies outside the circumsphere of the points in $V$, where the circumsphere is taken with respect to the norm induced by $Q$. If the sign is 0, then $\boldsymbol{w}$ lies on the circumsphere, and if it is negative, then $\boldsymbol{w}$ lies inside the circumsphere. In computational geometry this insphere/outsphere test is conveniently formulated using *oriented matroid* terminology (cf. [**41**, Chapter 1.8]): Let $V = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1})$ be affinely independent points in $\mathbb{R}^d$ with positive orientation and let $\boldsymbol{w} \in \mathbb{R}^d$. Then the *chirotope*

$$\chi_{(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{d+1},\boldsymbol{w})}(Q) = \operatorname{sign} \begin{vmatrix} 1 & \ldots & 1 & 1 \\ \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_{d+1} & \boldsymbol{w} \\ Q[\boldsymbol{v}_1] & \ldots & Q[\boldsymbol{v}_{d+1}] & Q[\boldsymbol{w}] \end{vmatrix}$$

satisfies $\chi_{(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{d+1},\boldsymbol{w})}(Q) = \operatorname{sign}\langle Q, N_{V,\boldsymbol{w}}\rangle$.

Proposition 4.10. *Let $P$ be a $d$-dimensional polyhedron in $\mathbb{R}^d$. Let $U$ be the linear subspace of all $Q \in \mathcal{S}^d$ satisfying $\langle N_{V,\boldsymbol{w}}, Q\rangle = 0$ for all affinely independent sets $V \subseteq \operatorname{vert} P$ of cardinality $d+1$ and for all $\boldsymbol{w} \in \operatorname{vert} P$. Let $F$ be a facet of $P$.*

(a) *Let $V$ and $V'$ be two sets of cardinality $d$ containing affinely independent vertices of $F$ and let $\boldsymbol{w} \in \operatorname{vert} P \setminus F$ and $\boldsymbol{w}' \in \mathbb{R}^d$. Then*

$$\pi_U(N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}) = \pi_U(N_{V'\cup\{\boldsymbol{w}\},\boldsymbol{w}'}).$$

(b) *Let $V$ be a set of cardinality $d$ containing affinely independent vertices of $F$ and let $\boldsymbol{u}, \boldsymbol{w} \in \operatorname{vert} P \setminus F$ and $\boldsymbol{w}' \in \mathbb{R}^d$. Then*

$$\pi_U(N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}) = \pi_U(N_{V\cup\{\boldsymbol{u}\},\boldsymbol{w}'}).$$

Proof. In both cases we will show that the difference of the two considered forms lies in the orthogonal complement of $U$.

(a) Every pair of affinely independent sets $V, V' \subseteq \operatorname{vert} F$ of cardinality $d$ can be connected by a chain $V = V_1, \ldots, V_n = V'$ of affinely independent sets $V_i \subseteq \operatorname{vert} F$ of cardinality $d$ such that $|V_i \cap V_{i+1}| = d-1$. So we can assume $|V \cap V'| = d-1$. Setting $\alpha_{\boldsymbol{w}'} = 1$ there exist unique numbers $\alpha_{\boldsymbol{w}}$ and $\alpha_{\boldsymbol{v}}$ for $\boldsymbol{v} \in V$ defining a affine dependency between the points $\boldsymbol{w}, \boldsymbol{w}'$ and $\boldsymbol{v}$ in $V$. This defines the form $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}$.

We define $\boldsymbol{v}_1$ by $\boldsymbol{v}_1 \in V \setminus V'$. Since the affine hull of $V$ and of $V'$ equals the affine hull of $F$ there exist numbers $\beta_{\boldsymbol{v}}$ for $\boldsymbol{v} \in V'$ such that $\sum_{\boldsymbol{v}\in V'} \beta_{\boldsymbol{v}} = 1$ and $\boldsymbol{v}_1 = \sum_{\boldsymbol{v}\in V'} \beta_{\boldsymbol{v}}\boldsymbol{v}$. Thus we have an affine dependency

$$\begin{aligned} 0 &= \alpha_{\boldsymbol{w}} + \alpha_{\boldsymbol{w}'} + \sum_{\boldsymbol{v}\in V\setminus\{\boldsymbol{v}_1\}} \alpha_{\boldsymbol{v}} + \sum_{\boldsymbol{v}\in V'} \alpha_{\boldsymbol{v}_1}\beta_{\boldsymbol{v}}, \\ \boldsymbol{0} &= \alpha_{\boldsymbol{w}}\boldsymbol{w} + \alpha_{\boldsymbol{w}'}\boldsymbol{w}' + \sum_{\boldsymbol{v}\in V\setminus\{\boldsymbol{v}_1\}} \alpha_{\boldsymbol{v}}\boldsymbol{v} + \sum_{\boldsymbol{v}\in V'} \alpha_{\boldsymbol{v}_1}\beta_{\boldsymbol{v}}\boldsymbol{v}, \end{aligned}$$

which defines the form $N_{V'\cup\{\boldsymbol{w}\},\boldsymbol{w}'}$. This gives

$$N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'} - N_{V'\cup\{\boldsymbol{w}\},\boldsymbol{w}'} = \alpha_{\boldsymbol{v}_1}\left(\boldsymbol{v}_1\boldsymbol{v}_1^t - \sum_{\boldsymbol{v}\in V'} \beta_{\boldsymbol{v}}\boldsymbol{v}\boldsymbol{v}^t\right).$$

Since $V \cup V'$ is a minimal affinely dependent set, we can choose an arbitrary vertex $\boldsymbol{w}$ of $P$ and find that the right hand side is a multiple of $N_{V'\cup\{\boldsymbol{w}\},\boldsymbol{v}_1}$.

(b) We take a closer look at the difference $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'} - N_{V\cup\{\boldsymbol{u}\},\boldsymbol{w}'}$ and show that it is a multiple of $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{u}}$. Set $\alpha_{\boldsymbol{w}'} = 1$ again and let $\alpha_{\boldsymbol{w}}$ and $\alpha_{\boldsymbol{v}}$ with $\boldsymbol{v} \in V$ be real numbers defining the affine dependency between the points $\boldsymbol{w}, \boldsymbol{w}'$ and $\boldsymbol{v}$ in $V$. This defines the form $N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'}$. In the same way let $\alpha'_{\boldsymbol{w}'} = 1$, $\alpha'_{\boldsymbol{u}}$ and $\alpha'_{\boldsymbol{v}}$ be real numbers defining $N_{V\cup\{\boldsymbol{u}\},\boldsymbol{w}'}$.

We set $\beta_{\boldsymbol{u}} = \alpha'_{\boldsymbol{u}}$, $\beta_{\boldsymbol{w}} = -\alpha_{\boldsymbol{w}}$ and $\beta_{\boldsymbol{v}} = \alpha'_{\boldsymbol{v}} - \alpha_{\boldsymbol{v}}$ for $\boldsymbol{v} \in V$. Then $\sum \beta_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{0}$ and $\sum \beta_{\boldsymbol{v}} = 0$ where the sums run through all $\boldsymbol{v} \in V \cup \{\boldsymbol{u}, \boldsymbol{w}\}$. Thus

$$N_{V\cup\{\boldsymbol{w}\},\boldsymbol{w}'} - N_{V\cup\{\boldsymbol{u}\},\boldsymbol{w}'} = \beta_{\boldsymbol{u}} N_{V\cup\{\boldsymbol{w}\},\boldsymbol{u}}. \tag{4.5}$$

□

Note, if $\boldsymbol{u}$ and $\boldsymbol{w'}$ in the last calculation lie in opposite halfspaces with respect to the affine plane through $V$, then $\beta_{\boldsymbol{u}} > 0$ in (4.5). Therefore, repeated application yields the following proposition, which we use for the proof of Theorem 4.7 and in Section 4.3.2 to prove Theorem 4.15.

PROPOSITION 4.11. *Let $V_1, \dots, V_m \subset \mathbb{R}^d$ be affinely independent sets of cardinality $d+1$ with $|V_i \cap V_{i+1}| = d$ for $i = 1, \dots, m-1$. Let $\boldsymbol{w} \in \mathbb{R}^d$ and $V_i$ be on opposite sides of* $\operatorname{aff}(V_i \cap V_{i+1})$ *for $i = 1, \dots, m-1$. Then*

$$N_{V_1,\boldsymbol{w}} = N_{V_m,\boldsymbol{w}} + \sum_{i=1}^{m-1} \alpha_i N_{V_i,\boldsymbol{v}_{i+1}}$$

*with $\boldsymbol{v}_{i+1} \in V_{i+1} \setminus V_i$ and positive constants $\alpha_i$, for $i = 1, \dots, m-1$.*

With these propositions at hand, we can give a proof of the "fundamental theorem".

PROOF OF THEOREM 4.7. I. We show that $\boldsymbol{\Delta}(\mathcal{D})$ is given by the set of listed linear equalities and inequalities. By (4.4) this implies that $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ is a polyhedral cone, because the Delone subdivision induced by a $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$ and a standard periodic vertex-set contains only finitely many Delone polyhedra up to $\mathbb{Z}^d$ invariant translations.

For $Q \in \boldsymbol{\Delta}(\mathcal{D})$ the linear equalities and inequalities are satisfied by Proposition 4.8.

Conversely, let us assume $Q \in \mathcal{S}^d$ satisfies the linear equalities and inequalities for every polytope in $\mathcal{D}$. By Proposition 4.8 and Proposition 4.10 we can assume that these are valid for all possible choices of $V$, $\boldsymbol{v}$ and $\boldsymbol{w}$. Note that for the use of Proposition 4.10 it is crucial to observe, that for two linear subspaces $U, U'$ of $\mathcal{S}^d$ with $T = U \cap U'$ we have $\pi_T = \pi_U \circ \pi_{U'} = \pi_{U'} \circ \pi_U$.

Let $P$ be a $d$-dimensional polyhedron in $\mathcal{D}$. Let $V \subseteq \operatorname{vert} P$ be the vertex-set of a $d$-simplex.

We show that all possible inequalities $\langle N_{V,w}, Q\rangle \geq 0$, where $\boldsymbol{w} \in \mathcal{D}$, are implied by the inequalities $\langle N_{V,\boldsymbol{v}}, Q\rangle \geq 0$, where $\boldsymbol{v} \in \mathcal{D}$ is either a vertex of $P$ or a vertex of an adjacent Delone polyhedron of $P$. Assume $\boldsymbol{w} \in \mathcal{D}\backslash P$. Then we choose a sequence of adjacent $d$-simplices with vertex-sets $V_1, \dots, V_m$ in $\mathcal{D}$ satisfying the requirements of Proposition 4.11. In addition we require that each vertex-set is contained in the vertex-set of a fixed polyhedron of $\mathcal{D}$, in particular $V_1 \subseteq \operatorname{vert} P$ and $\boldsymbol{w} \in V_m$. Note that this can be achieved by looking at a refining triangulation of $\mathcal{D}$, which we can choose arbitrarily. By our assumption on $Q$, we have $\langle N_{V_i,\boldsymbol{v}_{i+1}}, Q\rangle \geq 0$ with equality if and only if $V_i, V_{i+1}$ are subsets of the same polyhedron. Thus by Proposition 4.11 and Proposition 4.8 we see that the equation $\langle N_{V_m,\boldsymbol{w}}, Q\rangle = 0$ implies $\langle N_{V_1,\boldsymbol{w}}, Q\rangle > 0$.

Now we prove that a form $Q \in \mathcal{S}^d$ satisfying all linear conditions must be positive semidefinite. Let $\boldsymbol{w} \in \mathcal{D}$ and $\lambda \in \mathbb{Z}$ and write $\lambda \boldsymbol{w} = \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \boldsymbol{v}$ with $1 = \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}}$. Then

$$0 \leq \langle N_{V,\lambda\boldsymbol{w}}, Q\rangle = \lambda^2 Q[\boldsymbol{w}] - \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} Q[\boldsymbol{v}],$$

and hence $Q[\boldsymbol{w}] \geq 0$ since the coefficients $\alpha_{\boldsymbol{v}}$ depend only affinely on $\lambda$.

Furthermore, $Q$ lies in the rational closure $\tilde{\mathcal{S}}^d_{\geq 0}$: The linear conditions are rational and thus define a polyhedral cone in which every extreme ray contains rational semidefinite quadratic forms. By the argument in the proof of Proposition 4.1 all quadratic forms in such a polyhedral cone lie in $\tilde{\mathcal{S}}^d_{\geq 0}$.

We finally show that there exists a $\boldsymbol{c} \in \mathbb{R}^d$ and $r > 0$ such that $Q[\boldsymbol{w} - \boldsymbol{c}] \geq r^2$ with equality if and only if $\boldsymbol{w} \in \operatorname{vert} P$. A solution of the system of linear equations $Q[\boldsymbol{v} - \boldsymbol{c}] = Q[\boldsymbol{v}' - \boldsymbol{c}]$, where $\boldsymbol{v}, \boldsymbol{v}' \in V$ with $V \subseteq \operatorname{vert} P$, gives $\boldsymbol{c}$ and $r$. Applying Proposition 4.8 shows that the inequalities $Q[\boldsymbol{w} - \boldsymbol{c}] \geq r^2$ are valid with equality if and only if $\boldsymbol{w} \in \operatorname{vert} P$.

II. For the assertion on the isomorphism of posets given by the map $\mathcal{D} \mapsto \overline{\boldsymbol{\Delta}(\mathcal{D})}$, we need to verify that the Delone subdivision $\mathcal{D}$ is a true coarsening of $\mathcal{D}'$ if and only if $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ is strictly contained in $\overline{\boldsymbol{\Delta}(\mathcal{D}')}$. This follows from I. because if $\mathcal{D}$ is a true coarsening of $\mathcal{D}'$ we have equalities in the description of $\boldsymbol{\Delta}(\mathcal{D})$ which are inequalities in the description of $\boldsymbol{\Delta}(\mathcal{D}')$. If on the other hand, $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ is strictly contained in $\overline{\boldsymbol{\Delta}(\mathcal{D}')}$, we know that all equalities and inequalities in the description of $\boldsymbol{\Delta}(\mathcal{D}')$ are also satisfied by elements of $\boldsymbol{\Delta}(\mathcal{D})$ implying that $\mathcal{D}$ is a coarsening of $\mathcal{D}'$. Moreover, $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ has to be contained in the boundary of $\overline{\boldsymbol{\Delta}(\mathcal{D}')}$, because otherwise there would exist a $Q \in \boldsymbol{\Delta}(\mathcal{D}') \cap \boldsymbol{\Delta}(\mathcal{D})$ implying $\mathcal{D} = \mathcal{D}'$. Thus at least one of the inequalities in the description of $\overline{\boldsymbol{\Delta}(\mathcal{D}')}$ is fulfilled with equality for the elements of $\overline{\boldsymbol{\Delta}(\mathcal{D})}$. Hence $\mathcal{D}$ is a *true* coarsening of $\mathcal{D}'$.

Since every form in $\tilde{\mathcal{S}}^d_{\geq 0}$ defines a unique Delone subdivision of $\Lambda$, $\tilde{\mathcal{S}}^d_{\geq 0}$ is subdivided into secondary cones. This subdivision has the face-to-face property, hence is a polyhedral subdivision, due to the poset isomorphism: Otherwise we would find Delone subdivisions $\mathcal{D}$ and $\mathcal{D}'$, such that $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ and $\overline{\boldsymbol{\Delta}(\mathcal{D}')}$ have the same dimension and such that there is a common relative interior point, but with $\overline{\boldsymbol{\Delta}(\mathcal{D})} \neq \overline{\boldsymbol{\Delta}(\mathcal{D}')}$. Thus $Q$ would yield two different Delone subdivisions, which is not possible. $\square$

**4.2.3. Finiteness for rational periodic vertex-sets.** Theorem 4.7 generalizes Voronoi's Theorem 4.3 and Part I of his Theorem 4.5. Below we generalize Part II of Theorem 4.5 in case of rational standard periodic vertex-sets.

As in Section 3.2 we consider a standard periodic vertex-set

$$\Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d \tag{4.6}$$

determined by $\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{m-1}) \in \mathbb{R}^{d\times(m-1)}$ and $\boldsymbol{t}_m = \boldsymbol{0}$. Recall from Section 3.4.1 that the automorphism group of $\Lambda_{\boldsymbol{t}}$ is denoted by $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. By Proposition 3.19, $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ is a congruence subgroup (having finite index in $\mathsf{GL}_d(\mathbb{Z})$) if and only if $\boldsymbol{t}$ is rational. For rational $\boldsymbol{t}$ we obtain a finiteness result for the number of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone polytopes.

THEOREM 4.12. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$. Then there exist only finitely many $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone polytopes with respect to $\Lambda_{\boldsymbol{t}}$ (as in* (4.6)*) and some $Q \in \mathcal{S}^d_{>0}$.*

PROOF OF THEOREM 4.12. We loosely follow the proof of Deza, Grishukhin and Laurent in [**82**] for the classical case (see also [**83**]). Let $N$ denote the least common multiple of denominators in $\boldsymbol{t}$. Then we consider $\Lambda = N\Lambda_{\boldsymbol{t}} \subseteq \mathbb{Z}^d$. A Delone polytope $P$ (of some $Q \in \mathcal{S}^d_{>0}$ and with vertices in $\Lambda_{\boldsymbol{t}}$) is mapped to $NP$ with integral vertices, given by say $n$ vectors, which we join in a single matrix $V \in \mathbb{Z}^{d\times n}$.

Without loss of generality we may assume that $P$ is $d$-dimensional. Among the vertices of $NP$ we choose $d+1$ affinely independent ones spanning a simplex of maximum volume. We assume that the first $d+1$ columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ of $V$ contain these vertices. The differences $\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1} - \boldsymbol{v}_1$ form a basis of a sublattice of $\mathbb{Z}^d$. By a well known theorem in lattice theory (cf. [**51**, Section I.2.2]) there exists a $U \in \mathsf{GL}_d(\mathbb{Z})$ such that

$$M = U(\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1} - \boldsymbol{v}_1)$$

is an upper triangular, integral matrix $M = (m_{ij})_{i,j=1,\ldots,d}$ with $0 \leq m_{ij} < m_{ii}$ for all $i, j$ with $1 \leq i < j \leq d$.

Since the Delone subdivision of $\Lambda$ is invariant with respect to $N\mathbb{Z}^d$-invariant translations, we know that the simplex $S$, spanned by the first $d+1$ columns of $UV$, has volume

$$\operatorname{vol} S \leq \operatorname{vol}(NP) \leq \det(N\mathbb{Z}^d) = N^d.$$

On the other hand $\operatorname{vol} S = |\det M|/d!$. Hence $|\det M| \leq N^d d!$. In particular, there exist only finitely many matrices $M$, with the required properties.

The matrix $UV$ is not necessarily in the orbit $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})V$. But by choosing the unique $U' \equiv U^{-1} \bmod N$ with $U' \in \{0, \ldots, N-1\}^{d\times d}$, we map $V$ to $V' = U'UV = (\boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n) \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})V$. The first $d+1$ columns of $V'$ contain the vertices of a largest possible simplex (among those spanned by columns of $V'$). Because of translation invariance (with respect to $N\mathbb{Z}^d$) we may assume that the first column contains one of the $m$ scaled, integral translation vectors $N\boldsymbol{t}_i$. We consider the integral matrix

$$Y = (\boldsymbol{v}'_2 - \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n - \boldsymbol{v}'_1).$$

Its first $d$ columns contain one of finitely many possible matrices $D = U'M$.

We show that there exist only finitely many possible matrices $Y$, which proves that there are only finitely many $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone polytopes with $n$ vertices in $\Lambda_{\boldsymbol{t}}$. Consider $D^{-1}Y = (\mathrm{id}_d\, R)$ with $R = (r_{ij}) \in \mathbb{Q}^{d\times(n-(d+1))}$. By *Cramer's rule* we obtain $|r_{ij}| = |\det C|/|\det D|$ where $C$ is the matrix obtained from $D$ by replacing the $i$-th column of $D$ by the $(j+d+1)$-th column of $Y$. By the assumptions made on the first $d$ columns of $Y$, we know that $|\det C| \leq |\det D| \leq N^d d!$. (Note that $|\det D| = |\det M|$.) Thus there are only finitely many possible matrices $R$, respectively $Y$, with a fixed $n$.

It remains to show that the number of vertices $n$ of a Delone polytope with vertices in $\Lambda$ is bounded. A trivial bound is obtained as follows: We define an equivalence relation on $N\mathbb{Z}^d$: $\boldsymbol{v} \sim \boldsymbol{v}'$, if $\boldsymbol{v} + \boldsymbol{v}' \in 2N\mathbb{Z}^d$. No two vertices $\boldsymbol{v}$ and $\boldsymbol{v}'$ of $NP$ can belong to the same equivalence class, since otherwise, $(\boldsymbol{v} + \boldsymbol{v}')/2 \in N\mathbb{Z}^d \subseteq \Lambda$, which yields a contradiction to the definition of a Delone polytope. Hence $n \leq (2N)^d$. $\square$

Now we consider the secondary cones of Delone subdivisions with vertices in $\Lambda_{\boldsymbol{t}}$ and how the group $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ acts on them. We call two secondary cones $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$ $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent, if there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ with $\boldsymbol{\Delta}' = U^t \boldsymbol{\Delta} U$. For rational $\boldsymbol{t}$ the proof above yields a finiteness result for $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent secondary cones, in analogy not only to the classical theory, but also to the theory of $\boldsymbol{t}$-perfect forms in Section 3.4.

THEOREM 4.13. *Let $d, m \geq 1$ and $\Lambda_{\boldsymbol{t}}$ with $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$ be a standard periodic vertex-set as in* (4.6). *Then the group $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ acts by $\overline{\boldsymbol{\Delta}(\mathcal{D})} \mapsto U^t\overline{\boldsymbol{\Delta}(\mathcal{D})}U$ on the polyhedral subdivision of $\tilde{\mathcal{S}}^d_{\geq 0}$ into topological closures of secondary cones of Delone subdivisions $\mathcal{D}$ with vertex-set $\Lambda_{\boldsymbol{t}}$. If $\boldsymbol{t}$ is rational, then there exist only finitely many $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent secondary cones under this group action.*

PROOF. The first statement is a direct consequence of Theorem 4.7. The statement of the last sentence follows from an adaption of the proof of Theorem 4.12: We have to show that there exist only finitely many Delone subdivisions $\operatorname{Del}_{\Lambda_{\boldsymbol{t}}}(Q)$ up to $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalence. Every Delone subdivision is determined by the Delone polytopes containing one of the vertices $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_m$. Their number and the number of their vertices is finite, following for example from a volume argument, as in the proof of Theorem 4.12. Choosing a simplex of maximum volume contained in one of the Delone polytopes, allows the same argumentation as in the proof there. $\square$

We think for non-rational $\boldsymbol{t}$ it can be shown that there exist infinitely many pairwise $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone polytopes with vertices in $\Lambda_{\boldsymbol{t}}$.

**4.2.4. Other generalizations of Voronoi's reduction theory.** Our generalization of Voronoi's theory is not the first and only one. Due to a demand in algebraic geometry people have studied several others.

In the seventies, Mumford and Satake described a general procedure for *compactifying* quotients of *bounded symmetric domains* by *arithmetic groups* (cf. [**5**]). Their constructions use special decompositions of the domains of interest into rational polyhedra. As an application of this general method, Namikawa ([**186**]) constructed a compactification of *Siegel modular varieties* using Voronoi's reduction theory. Siegel modular varieties are moduli spaces of Abelian varieties. Over the complex numbers they arise as quotient of the Siegel upper halfspace by an arithmetic group. Namikawa's construction opened the possibility to understand the boundary of the moduli space with help of degenerations. Later algebraic geometers refined this work and they used also generalizations of Voronoi's reduction theory for this. In this section we want to review these generalizations and compare them to ours. For more information about the geometry of Siegel modular varieties we refer to the survey [**142**] of Hulek and Sankaran.

Oda and Seshadri study in [**191**] a generalization of Delone subdivisions which they call Namikawa decompositions: Let $E$ be a Euclidean space where an orthogonal decomposition into subspaces $E'$ and $E''$ is given: $E = E' \perp E''$, and let $\Lambda \subseteq E$ be a lattice so that $\Lambda \cap E'$ is also a lattice. For a vector $\psi \in E''$ the Namikawa decomposition is

$$D_\psi = \{\pi'(D(x)) : x \in E' + \psi\},$$

where $\pi' : E \to E'$ is the orthogonal projection onto $E'$ and

$$D(x) = \operatorname{conv}\{\xi_1, \ldots, \xi_r\},$$

with vertices $\xi_i \in \Lambda$ closest to $x$, is a Delone polytope of $\Lambda$. Then, Oda and Seshadri explain in [**191**, Proposition 2.3] how the Namikawa decomposition changes when one varies the vector $\psi$ along $E''$. Our generalization of Voronoi's reduction theory goes into a different direction: We fix a lattice $\Lambda$ in a real vector space of finite dimension and a basis of $\Lambda$ and study how the Delone subdivision of $\Lambda$ changes when one varies the inner product along a given subspace (in the space of Gram matrices).

Alexeev and collaborators (cf. the survey [**1**, Section 5]) consider so-called semi-Delaunay decompositions: Let $M \subseteq \mathbb{R}^d$ be a lattice and $\Gamma \subseteq M$ be a sublattice of finite index. A $\Gamma$-periodic polyhedral subdivision of $\mathbb{R}^d$ is called a semi-Delaunay decomposition when it is the projection of the lower hull (see Section 4.3.2) of the lifted points $(m, h(m))$ with $m \in M$ and $h : M \to \mathbb{R}$ a function of the form $h(m) = q(m) + r(m)$ where $q : \mathbb{R}^d \to \mathbb{R}$ is a positive semidefinite quadratic form and $r : M/\Gamma \to \mathbb{R}$ an arbitrary function on the cosets. Note that Delone subdivisions of rational standard periodic sets, i.e. sets of the form (4.1) with rational $t_i'$, are covered by this construction: Take $\Gamma = \mathbb{Z}^d$ and an overlattice $M$ containing all $t_i'$. Define $r(t_i' + \Gamma) = 0$ and $r(m + \Gamma)$ large enough for all $m \in M$ not in the rational standard periodic set. Nevertheless, our treatment has its merits. It is very explicit and, as we show in this book, can immediately be used for computations. Moreover, our generalization of Voronoi's reduction theory to positive definite quadratic forms with prescribed automorphism group that follows (and which is essential for our applications) is not covered by the reviewed constructions.

## 4.3. Equivariant extension and $T$-secondary cones

**4.3.1. $T$-secondary cones.** In this section we generalize Voronoi's second reduction theory further, analogously to the theory of $T$-perfect forms described in Section 3.4.4. We consider a linear subspace $T \subseteq \mathcal{S}^d$ and look at *$T$-secondary cones* of Delone subdivisions $\mathcal{D}$ defined by

$$\mathbf{\Delta}_T(\mathcal{D}) = \mathbf{\Delta}(\mathcal{D}) \cap T.$$

We call a $T$-secondary cone $\mathbf{\Delta}_T(\mathcal{D})$ and the corresponding Delone subdivision $\mathcal{D}$ *$T$-generic* if $\dim \mathbf{\Delta}_T(\mathcal{D}) = \dim T$. If the vertex-set of $\mathcal{D}$ is a standard periodic set, then by Theorem 4.7 the topological closures of $T$-generic, $T$-secondary cones give a polyhedral subdivision of $\tilde{\mathcal{S}}^d_{\geq 0} \cap T$. Two $T$-generic cones are called *contiguous* if their closures share a facet. A difference with the classical theory is the existence of *dead-ends*, which are facets only incident to one $T$-generic cone. These necessarily contain only non-positive forms in $\tilde{\mathcal{S}}^d_{\geq 0} \setminus \mathcal{S}^d_{>0}$.

The ultimate goal would be to state for every subspace $T$ a theorem as Part II of Theorem 4.5 (which deals with the case $T = \mathcal{S}^d$ and vertex-set $\mathbb{Z}^d$). It turns out though that in general, this is not always possible. If $\dim T = 1$, the intersection of $T$ with $\mathcal{S}^d_{>0}$ contains a PQF $Q$ and all its positive multiples. In this case a generalized Theorem 4.5 is trivially true. Therefore, if not stated otherwise, we assume $\dim T \geq 2$ in what follows.

The "road map" for our generalization is the following: In Section 4.3.2 we generalize the notion of bistellar neighbors to the new setting. In the new theory not every subspace $T$ gives a polyhedral subdivision of $\tilde{\mathcal{S}}^d_{\geq 0} \cap T$ with only finitely many inequivalent $T$-secondary cones (see the examples in Section 4.3.3). For specific vertex-sets and subspaces though, there exist only finitely many inequivalent $T$-secondary cones. We give such a finiteness result in Section 4.3.3, for rational standard periodic vertex-sets $\Lambda_{\boldsymbol{t}}$ and linear subspaces $T = T_G$ containing all $G$-invariant PQFs for a given finite subgroup $G$ of $\mathsf{GL}^{\boldsymbol{t}}_d(\mathbb{Z})$. Thus we obtain a generalized, equivariant version of Voronoi's reduction theory.

**4.3.2. Flips and bistellar neighbors.** Given a linear subspace $T$ of $\mathcal{S}^d$, we know by Theorem 4.7 that $\tilde{\mathcal{S}}^d_{\geq 0} \cap T$ is covered by the topological closures of $T$-generic $T$-secondary cones. Throughout this section we assume that the vertex-set of the considered Delone subdivisions is a fixed standard periodic set $\Lambda$. Given a $T$-generic Delone subdivision $\mathcal{D}$ and its $T$-secondary cone $\boldsymbol{\Delta}_T(\mathcal{D})$ we want to determine its contiguous $T$-secondary cones. For this let $\mathcal{F}$ be a facet of $\overline{\boldsymbol{\Delta}_T(\mathcal{D})}$ which is not a dead-end. Let $\mathcal{D}'$ be the $T$-generic Delone subdivision with $\boldsymbol{\Delta}(\mathcal{D}')$ being contiguous to $\boldsymbol{\Delta}(\mathcal{D})$ at $\mathcal{F}$, hence with $\mathcal{F} = \overline{\boldsymbol{\Delta}_T(\mathcal{D})} \cap \overline{\boldsymbol{\Delta}_T(\mathcal{D}')}$. The transition from $\mathcal{D}$ to $\mathcal{D}'$ is called a *$T$-flip*; the subdivisions $\mathcal{D}$ and $\mathcal{D}'$ are referred to as *bistellar neighbors*.

In order to describe the $T$-flip, let $N_{\mathcal{F}} \in \mathcal{S}^d$ denote the form which is uniquely determined by the following conditions:

(i) $N_{\mathcal{F}} \in T$ with $\langle N_{\mathcal{F}}, N_{\mathcal{F}} \rangle = 1$,
(ii) $\langle N_{\mathcal{F}}, Q \rangle = 0$ for all $Q \in \mathcal{F}$,
(iii) $\langle N_{\mathcal{F}}, Q \rangle > 0$ for all $Q \in \boldsymbol{\Delta}_T(\mathcal{D})$,
(iv) $\langle N_{\mathcal{F}}, Q \rangle < 0$ for all $Q \in \boldsymbol{\Delta}_T(\mathcal{D}')$.

We collect facets $F$ of $\mathcal{D}$ whose forms $N_{\mathcal{D},F}$ (as defined after Theorem 4.7) are positive multiples of $N_{\mathcal{F}}$ when projected onto $T$:

$$\mathcal{R}_{\mathcal{F}} = \{F \in \mathcal{D} : \dim F = d-1 \text{ and } N_{\mathcal{F}} = \alpha \cdot \pi_T(N_{\mathcal{D},F}) \text{ for a } \alpha > 0\}.$$

On $\mathcal{R}_{\mathcal{F}}$ we define an equivalence relation by

$$F \sim F' \iff \begin{cases} \text{there exist } F_1, \ldots, F_n \in \mathcal{R}_{\mathcal{F}} \text{ with } F = F_1,\ F' = F_n, \\ \text{and } d\text{-dimensional polyhedra } P_0, \ldots, P_n \in \mathcal{D} \\ \text{with } F_i = P_{i-1} \cap P_i \text{ for } i = 1, \ldots, n. \end{cases} \tag{4.7}$$

Thus by definition, the facets in each equivalence class are connected by a chain of adjacent $d$-dimensional polyhedra in $\mathcal{D}$. The union of these polyhedra is a polyhedron again:

Proposition 4.14. *Let $\mathcal{D}$ be a Delone subdivision and $\mathcal{C}$ an equivalence class of* (4.7). *Let $V_{\mathcal{C}}$ be the set of all vertices of $d$-dimensional polytopes in $\mathcal{D}$ with a facet in $\mathcal{C}$. Then* $(\operatorname{conv} V_{\mathcal{C}}) \cap \Lambda = V_{\mathcal{C}}$.

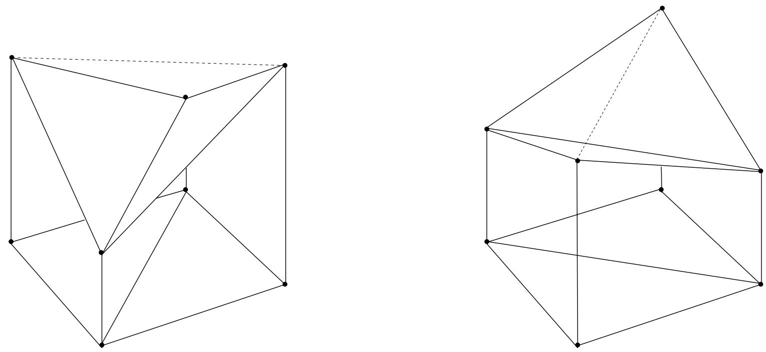

FIGURE 1. Lifting interpretation of "upper to lower hull change".

A proof of the proposition is given at the end of this section. It, as well as the description of $T$-flips can conveniently be given using the *lifting map* $w_Q : \mathbb{R}^d \to \mathbb{R}^{d+1}$ defined by $w_Q(\boldsymbol{x}) = (\boldsymbol{x}, Q[\boldsymbol{x}])$ for a $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$. The polyhedra $\operatorname{conv} V_{\mathcal{C}}$ in Proposition 4.14 are called *repartitioning polyhedra* of $\mathcal{F}$. For a $Q \in \boldsymbol{\Delta}_T(\mathcal{D})$ and a repartitioning polyhedron $P$ we define the $(d+1)$-dimensional polyhedron

$$\bar{P}(Q) = \operatorname{conv}\{w_Q(\boldsymbol{v}) : \boldsymbol{v} \in \operatorname{vert} P\}.$$

For each of its facets $\bar{F}$ we have an outer normal vector $\boldsymbol{n}(\bar{F}) \in \mathbb{R}^{d+1}$. The facets are divided into three groups. We speak of a *lower facet*, if the last coordinate of the normal vector satisfies $(\boldsymbol{n}(\bar{F}))_{d+1} < 0$; we speak of an *upper facet*, if $(\boldsymbol{n}(\bar{F}))_{d+1} > 0$ and of a *lateral facet*, if $(\boldsymbol{n}(\bar{F}))_{d+1} = 0$. We show below that these notions are independent of the particular choice of $Q \in \boldsymbol{\Delta}_T(\mathcal{D})$.

For the description of the $T$-flip in the following theorem, let $\pi : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ denote the projection onto the first $d$ coordinates. Figure 1 gives an example for the change from "lower to upper hull" in the case of a planar repartitioning polytope. Note that the situation in higher dimensions can be much more complicated than what the picture might suggest.

THEOREM 4.15. *Let $T$ be a linear subspace of $\mathcal{S}^d$ and $\mathcal{F}$ be a common facet of two contiguous $T$-secondary cones of $T$-generic Delone subdivisions $\mathcal{D}$ and $\mathcal{D}'$. Then we obtain the d-dimensional polytopes of $\mathcal{D}'$ from those of $\mathcal{D}$ by choosing an arbitrary $Q \in \boldsymbol{\Delta}(\mathcal{D})$ and by doing the following for each repartitioning polytope $P$ of $\mathcal{F}$:*

(i) *We remove all polytopes $\pi(\bar{F})$ of lower facets $\bar{F}$ of $\bar{P}(Q)$.*
(ii) *We add all polytopes $\pi(\bar{F})$ of upper facets $\bar{F}$ of $\bar{P}(Q)$.*

REMARK 4.16. Our theory is similar to the theory of equivariant secondary polytopes of regular polytopal subdivisions, as recently described by Reiner in [**201**]. With slight modifications our explicit description of the flipping procedure works in his setting as well.

As a preparation for the proofs of Theorem 4.15 and Proposition 4.14, consider the set

$$\operatorname{conv}\{w_Q(\boldsymbol{x}) : \boldsymbol{x} \in \Lambda\}. \tag{4.8}$$

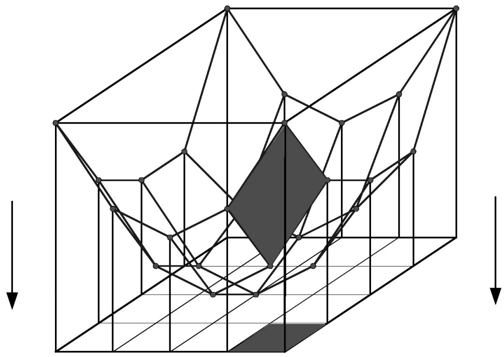

FIGURE 2. Delone subdivision obtained from lifting.

It is a locally finite polyhedron. A first and important observation is that the facets of the polyhedron (4.8) yield the $d$-polytopes of the Delone subdivision when projected by $\pi$ onto the first $d$ coordinates (see Figure 2, cf. [**98**]):

PROPOSITION 4.17. *Let $\Lambda$ be a standard periodic set and $Q \in \tilde{\mathcal{S}}^d_{\geq 0}$. Then*

$$\mathrm{Del}_\Lambda(Q) = \{\pi(\bar{F}) : \bar{F} \textit{ face of } (4.8) \textit{ of dimension less than or equal } d\}.$$

PROOF. For a polytope $P \in \mathcal{D}$ and a set $V \subseteq \operatorname{vert} P$ of cardinality $d+1$ and with affinely independent vertices, the set of all $(\boldsymbol{x}, y) \in \mathbb{R}^{d+1}$ with

$$y = Q[\boldsymbol{x}] - \langle N_{V,\boldsymbol{x}}, Q\rangle = \sum_{\boldsymbol{v} \in V} \alpha_{\boldsymbol{v}} Q[\boldsymbol{v}] \tag{4.9}$$

is an affine hyperplane in $\mathbb{R}^{d+1}$. Recall from the definition of $N_{V,\boldsymbol{x}}$ in (4.3) that the $\alpha_{\boldsymbol{v}}$ are uniquely defined by the affine dependency $\boldsymbol{x} = \sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \boldsymbol{v}$ with $\sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} = 1$. Thus we see that the hyperplane given by (4.9) contains the $d+1$ lifted points $w_Q(\boldsymbol{v})$. Moreover, it is a supporting hyperplane of the polyhedron (4.8) if and only if $\langle N_{V,\boldsymbol{x}}, Q\rangle \geq 0$ for all $\boldsymbol{x} \in \Lambda$. By Proposition 4.8 this is the case if and only if $Q[\boldsymbol{x} - \boldsymbol{c}] \geq r^2$ for a suitable $\boldsymbol{c} \in \mathbb{R}^d$ and $r > 0$. Here, equality holds if and only if $\boldsymbol{x} \in \operatorname{vert} P$. $\square$

The hyperplane given by equation (4.9) in the proof is a different one for subsets $V$ and $V'$ of different Delone polytopes. The forms $N_{\mathcal{D},F}$ relate the change for adjacent $d$-polytopes $P$ and $P'$ in $\mathcal{D}$ with common facet $F = P \cap P'$. If the segment connecting $\boldsymbol{x}$ with a vertex $\boldsymbol{v}$ of $P$ intersects facets $F_1, \ldots, F_n$ of $\mathcal{D}$, then by Proposition 4.11 and by the definition of $N_{\mathcal{D},F}$ (after and because of Theorem 4.7) we have

$$y = Q[\boldsymbol{x}] - \langle N_{V,\boldsymbol{x}}, Q\rangle = Q[\boldsymbol{x}] - \sum_{i=1}^{n} \beta_i \langle N_{\mathcal{D},F_i}, Q\rangle \tag{4.10}$$

with suitable positive constants $\beta_i$, depending on $\boldsymbol{x}$.

Equation (4.10) yields simple proofs for Proposition 4.14 and Theorem 4.15:

PROOF OF PROPOSITION 4.14. Due to (4.10) and the definition of $\mathcal{C}$, all points $w_Q(\boldsymbol{v})$ with $\boldsymbol{v} \in V_{\mathcal{C}}$ are coplanar in $\mathbb{R}^{d+1}$. Suppose the union of the $d$-dimensional polyhedra of $\mathcal{D}$ with a facet in $\mathcal{C}$ is strictly contained in $\operatorname{conv} V_{\mathcal{C}}$. Then there exists a facet $F$ of one of the polyhedra such that $F \notin \mathcal{C}$ and $F$ does not belong to the boundary of $\operatorname{conv} V_{\mathcal{C}}$. Hence, there exist two vertices $\boldsymbol{w}, \boldsymbol{w}' \in V_{\mathcal{C}}$ in opposite halfspaces with respect to $\operatorname{aff} F$. Then by (4.10) and since $\langle N_{\mathcal{D},F}, Q\rangle > 0$ we see that $w_Q(\boldsymbol{w})$, $w_Q(\boldsymbol{w}')$ and $w_Q(\boldsymbol{v})$ with $\boldsymbol{v} \in F$ cannot be coplanar, which is a contradiction. $\square$

Proof of Theorem 4.15. Choose $Q \in \boldsymbol{\Delta}(\mathcal{D})$ and $Q' \in \boldsymbol{\Delta}(\mathcal{D}')$ and let $P$ be a repartitioning polytope of $\mathcal{F} = \overline{\boldsymbol{\Delta}(\mathcal{D})} \cap \overline{\boldsymbol{\Delta}(\mathcal{D}')}$. Then by formula (4.10) and since $N_{\mathcal{D},F} = -N_{\mathcal{D}',F}$ for all facets $F$ contained in $P$, the upper facets of $\bar{P}(Q)$ project to the same polyhedral subdivision of $P$ as the lower facets of $\bar{P}(Q')$ and vice versa. The lower facets give the corresponding Delone subdivisions by Proposition 4.17, which proves the assertion. $\square$

**4.3.3. Equivariance and finiteness.** Let $T$ be a linear subspace of $\mathcal{S}^d$. We say that two $T$-secondary cones $\boldsymbol{\Delta}_T$ and $\boldsymbol{\Delta}'_T$ are *$T$-equivalent* if there is a $U \in \mathsf{GL}_d(\mathbb{Z})$ such that $U^t \boldsymbol{\Delta}_T U = \boldsymbol{\Delta}'_T$ and $U^t T U = T$. In case $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ we speak of *$(T, \boldsymbol{t})$-equivalence*. Otherwise we say that they are *$T$-inequivalent*, respectively *$(T, \boldsymbol{t})$-inequivalent*.

In this section we discuss assumptions on $T$ which ensure that there exist only finitely many, $T$-inequivalent or $(T, \boldsymbol{t})$-inequivalent $T$-secondary cones. Note that Voronoi's classical theory (cf. Theorem 4.5) deals with the case $T = \mathcal{S}^d$.

The main difference with the classical theory is that for general $T$ there may be infinitely many, $T$-inequivalent $T$-secondary cones:

Example 4.18. Consider a rational PQF $Q$ with trivial automorphism group. The orbit of $Q$ under the action of $\mathsf{GL}_d(\mathbb{Z})$ contains only rational PQFs. We choose a subspace $T$ through $Q$ in which the only rational forms are multiples of $Q$. So $T$ does not contain another element from the orbit of $Q$. Thus the setwise stabilizer of $T$ is trivial. In this situation there are infinitely many $T$-generic secondary cones whenever the set $T \cap \tilde{\mathcal{S}}^d_{\geq 0}$ is not closed in $\mathcal{S}^d$. Then the intersection of $T$ with the boundary of $\tilde{\mathcal{S}}^d_{\geq 0}$ is not covered by finitely many $T$-dead-ends. The maximum dimension of the intersection of $\mathcal{S}^d_{\geq 0}$ with a supporting hyperplane of $\mathcal{S}^d_{\geq 0}$ is $\binom{d+1}{2} - d$ (see [**217**, §10]). Thus the intersection of $T$ with the boundary of $\tilde{\mathcal{S}}^d_{\geq 0}$ cannot be covered by $T$-dead-ends if $\dim(T \cap \mathcal{S}^d_{\geq 0}) > \binom{d+1}{2} - d + 1$. In particular for all $d \geq 3$ there exist examples of subspaces $T$ with infinitely many, inequivalent $T$-generic cones.

Of particular interest is the case where the subspace $T = T_G$ is pointwise stabilized by a finite subgroup $G$ of $\mathsf{GL}_d(\mathbb{Z})$ respectively of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. We speak of $G$-secondary cones and $G$-equivalence, respectively of $(G, \boldsymbol{t})$-secondary cones and $(G, \boldsymbol{t})$-equivalence in this case. Lemma 3.26 together with Theorem 4.13 give the following theorem, which finishes our equivariant version of Voronoi's second reduction theory.

Theorem 4.19. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{Q}^{d \times (m-1)}$. Let $G$ be a finite subgroup of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. Then there exist only finitely many $(G, \boldsymbol{t})$-inequivalent $(G, \boldsymbol{t})$-secondary cones.*

Note that the case of $T$ being pointwise stabilized by a finite subgroup of $\mathsf{GL}_d(\mathbb{Z})$ is not the only case when there are only finitely many $T$-inequivalent $T$-secondary cones. Bayer–Fluckiger and Nebe ([**35**, Theorem 3.1]) give another sufficient condition for $T$ which ensures finiteness in the two-dimensional case. However, we are not aware of further, general results regarding this question. Note though that every computation that terminates is a particular result. For all the subspaces $T$ we considered for the applications, there were only finitely many $T$-inequivalent $T$-secondary cones. These subspaces are all spanned by rational forms, but this alone does not guarantee finiteness for $d \geq 3$, as shown by the following example due to Yves Benoist (private communication):

EXAMPLE 4.20. Consider the subspace $T$ spanned by $x^2 + 2y^2 + z^2$ and $xy$. It contains the two positive semidefinite, but non-positive forms $(x \pm \sqrt{2}y)^2 + z^2$. In any small neighborhood of these two forms, we find infinitely many forms in $T \cap \mathcal{S}^d_{>0}$ with pairwise differing Delone subdivisions. In particular, the minimal non-zero vectors $\boldsymbol{v} \in \mathbb{Z}^d$ tend towards the kernel of one of the two forms; the Delone subdivisions have differing edges $\operatorname{conv}\{\mathbf{0}, \boldsymbol{v}\}$ with $\boldsymbol{v} = (p, q, 0)^t$ and $\frac{p}{q}$ sufficiently close to $\mp\sqrt{2}$. Thus there are infinitely many different $T$-secondary cones, but the setwise stabilizer of $T$ in $\mathsf{GL}_d(\mathbb{Z})$ is finite. In order to see this, note that the three lines of rank-2 forms in $T$ have to be permuted. Note that the same argument does not apply to dimension 2. The space spanned by $x^2 + 2y^2$ and $xy$ has no finite setwise stabilizer.

**4.3.4. Algorithmic issues.** In this section we discuss several algorithmic issues which have to be resolved to turn our generalization of Voronoi's reduction theory into an effective procedure. We assume that all Delone subdivisions are with respect to a given, rational periodic vertex-set $\Lambda = \Lambda_{\boldsymbol{t}}$, with $\boldsymbol{t} \in \mathbb{Q}^{d \times (m-1)}$ (see (4.6)). The case $m = 1$ can be read as the lattice case where $\Lambda$ is $\mathbb{Z}^d$.

We emphasize those points where the generalization differs from the classical case discussed in Section 4.1. First we adapt Algorithm 3, which enumerates all inequivalent Delone triangulations. Algorithm 4 describes a procedure to enumerate all $T$-inequivalent $T$-generic Delone subdivisions. Depending on $T$ (see Section 4.3.3) this generalization may not terminate after finitely many steps (so it may actually not be an algorithm).

Apart from this, when we compare Algorithm 4 with Algorithm 3 of the original theory, essentially two new tasks arise:

(1) We have to find an initial $T$-generic Delone subdivision (cf. Algorithm 5), whereas in the classical case this can be given by "Voronoi's first subdivision".
(2) We have to check whether two $T$-generic Delone subdivisions are $(T, \boldsymbol{t})$-equivalent (cf. Algorithm 6).

For Algorithm 5 we need to compute the Delone subdivision of a PQF $Q$ and a (standard periodic) vertex-set $\Lambda$. We refer to [**98**] as a general reference (cf. [**92**]). In Algorithm 5 we need to compute the *$T$-rigidity index* of a Delone subdivision. It is simply the dimension of the linear span of $\boldsymbol{\Delta}_T(\mathcal{D})$. In the classical setting $T = \mathcal{S}^d$ the *rigidity index* was introduced by Baranovskii and Grishukhin in [**20**]. Algorithm 5 is a *randomized algorithm* and belongs to the class of so-called *Las Vegas algorithms*. That is, it always produces correct results whereas the running time is a random variable.

**Input:** Linear subspace $T \subseteq \mathcal{S}^d$ and a finite subgroup $G$ of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$, $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$
**Output:** Set $\mathcal{R}$ of all $T$-inequivalent $T$-generic Delone subdivision with vertex-set $\Lambda_{\boldsymbol{t}}$
$Y \leftarrow \{\mathcal{D}_1\}$, where $\mathcal{D}_1$ is a $T$-generic Delone subdivision (Algorithm 5).
$\mathcal{R} \leftarrow \emptyset$.
**while** there is a $\mathcal{D} \in Y$ **do**
  $Y \leftarrow Y\backslash\{\mathcal{D}\}$. $\mathcal{R} \leftarrow \mathcal{R} \cup \{\mathcal{D}\}$.
  compute the linear inequalities of $\overline{\boldsymbol{\Delta}_T(\mathcal{D})}$ as in Theorem 4.7.
  compute the facets $F_1, \dots, F_n$ of $\overline{\boldsymbol{\Delta}_T(\mathcal{D})}$.
  **for** $i = 1, \dots, n$ **do**
    **if** $F_i$ is not a dead-end **then**
      compute bistellar neighbor $\mathcal{D}_i$ of $\mathcal{D}$, defined by $F_i$ as in Theorem 4.15.
      **if** $\mathcal{D}_i$ is not $(T, \boldsymbol{t})$-equivalent to a Delone subdivision in the set
        $\mathcal{R} \cup \{\mathcal{D}_j : j \in \{1, \dots, i-1\},\ F_j \text{ not a dead-end}\}$ (Algorithm 6) **then**
        $Y \leftarrow Y \cup \{\mathcal{D}_i\}$.
      **end if**
    **end if**
  **end for**
**end while**

ALGORITHM 4. Enumeration of all $T$-inequivalent $T$-generic Delone subdivisions.

**Input:** Linear subspace $T \subseteq \mathcal{S}^d$ and a standard periodic vertex-set $\Lambda$
**Output:** A $T$-generic Delone subdivision $\mathcal{D}_1$
**repeat**
  Choose a random $Q$ in $T \cap \mathcal{S}^d_{>0}$.
  Compute $\mathcal{D}_1 = \mathrm{Del}_\Lambda(Q)$.
  Compute $T$-rigidity index $m$ of $\mathcal{D}_1$ by the linear equalities in Theorem 4.7.
**until** $m = \dim T$.

ALGORITHM 5. Finding a $T$-generic Delone subdivision.

Algorithm 6 checks whether two $T$-generic $(T, \boldsymbol{t})$-secondary cones are $(T, \boldsymbol{t})$-equivalent. For this we need an additional definition. We say that a positive semidefinite quadratic form $Q$ is *rational normalized* if it is integral and the entries have greatest common divisor equal to one.

For Algorithm 6 we have to test whether two PQFs are $(T, \boldsymbol{t})$-equivalent and we have to compute the $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-restricted automorphism group $\mathrm{Aut}_{\boldsymbol{t}}(Q)$ of a given PQF $Q$. For both tasks we refer to the discussion in Section 3.4.4.

The closed polyhedral cones $\overline{\boldsymbol{\Delta}_T(\mathcal{D})}$ are *rational* assuming that the vertex-set $\Lambda = \Lambda_{\boldsymbol{t}}$ is rational, respectively that $\boldsymbol{t}$ is rational. Thus these polyhedral cones have a description by rational inequalities as well as a description $\overline{\boldsymbol{\Delta}_T(\mathcal{D})} = \mathrm{cone}\{R_1, \dots, R_k\}$ as a cone generated by finitely many rational normalized quadratic forms lying in extreme rays of $\overline{\boldsymbol{\Delta}_T(\mathcal{D})}$.

Dealing with symmetries of a polyhedral cone in $\mathcal{S}^d$, the *characteristic form* $Q = \sum_{i=1}^k R_i$ is of great importance. This is due to the following lemma which is

**Input:** Linear subspace $T \subseteq \mathcal{S}^d$, and $T$-generic Delone subdivisons $\mathcal{D}_1$, $\mathcal{D}_2$ (w.r.t. $\Lambda_{\boldsymbol{t}}$)
**Output:** Yes, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are $(T, \boldsymbol{t})$-equivalent; No, otherwise
$\mathcal{R}_i \leftarrow \{$rational normalized $R$ of extreme rays of $\overline{\boldsymbol{\Delta}_T(\mathcal{D}_i)}\}$, for $i = 1, 2$.
$Q_i \leftarrow \sum_{R \in \mathcal{R}_i} R$, for $i = 1, 2$.
Compute $\mathrm{Aut}_{\boldsymbol{t}}(Q_1)$.
**if** there exists $A \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ and $B \in \mathrm{Aut}_{\boldsymbol{t}}(Q_1)$
 with $A^t Q_1 A = Q_2$ and $(BA)^t T(BA) = T$ **then**
 **return** Yes.
**else**
 **return** No.
**end if**

ALGORITHM 6. Checking $T$-equivalence.

implicitly used in Algorithm 6. Two rational polyhedral cones $\mathcal{P}$ and $\mathcal{P}'$ in $\mathcal{S}^d$ are called $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-*equivalent*, if there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ such that $U^t \mathcal{P} U = \mathcal{P}'$. As in the proof of Lemma 3.26, the *automorphism group* of $\mathcal{P}$ is

$$\mathrm{Aut}_{\boldsymbol{t}}(\mathcal{P}) = \{U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) : U^t \mathcal{P} U = \mathcal{P}\}.$$

LEMMA 4.21. *Let $d \geq 2$ and let $\mathcal{P}$ be a polyhedral cone in $\tilde{\mathcal{S}}^d_{\geq 0}$. Let $\boldsymbol{t} \in \mathbb{Q}^{d\times(m-1)}$. Then the automorphism group $\mathrm{Aut}_{\boldsymbol{t}}(Q)$ of the characteristic form $Q$ of $\mathcal{P}$ contains the automorphism group $\mathrm{Aut}_{\boldsymbol{t}}(\mathcal{P})$. The characteristic forms of two $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent rational polyhedral cones are equivalent.*

PROOF. Suppose that for two polyhedral cones $\mathcal{P}, \mathcal{P}' \subset \tilde{\mathcal{S}}^d_{\geq 0}$ (possibly the same) there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ such that $U^t \mathcal{P} U = \mathcal{P}'$. Extreme rays of $\mathcal{P}$ are mapped onto extreme rays of $\mathcal{P}'$. We need to show that this is also true for the uniquely determined rational normalized elements of the extreme rays. For this let $Q$ be such a rational normalized element of an extreme ray of $\mathcal{P}$. It is mapped onto the integral matrix $U^t Q U = \alpha Q'$ defining an extreme ray of $\mathcal{P}'$, with rational normalized $Q'$. Thus $\alpha \in \mathbb{N}$. On the other hand $Q'$ is mapped onto an integral multiple of $Q$ via $U^{-1}$ and $(U^{-1})^t Q' (U^{-1}) = (1/\alpha) Q$. Thus $\alpha = 1$, which proves the assertion. ☐

## 4.4. More on Delone polytopes and complexes

**4.4.1. Secondary cones of polyhedral complexes.** A natural question in the context of Voronoi's theory and its extensions is what kind of Delone polyhedra (with respect to some $Q \in \mathcal{S}^d_{\geq 0}$) can occur for a given discrete vertex-set $\Lambda \subset \mathbb{R}^d$. We define the *secondary cone* of a polyhedron $P$ with $\mathrm{vert}\, P \subset \Lambda$ by

$$\boldsymbol{\Delta}(\Lambda, P) = \{Q \in \mathcal{S}^d_{\geq 0} : P \in \mathrm{Del}_\Lambda(Q)\}.$$

Clearly, a necessary condition for $P = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots\}$ to have a nonempty secondary cone $\boldsymbol{\Delta}(\Lambda, P)$, is that its vertices $\mathrm{vert}\, P = \{\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots\}$ are all the points in $P \cap \Lambda$.

From Proposition 4.8 one derives that for some arbitrarily chosen, affinely independent $V \subseteq \operatorname{vert} P$ of cardinality $\dim \operatorname{aff} P + 1$ we have

$$\begin{aligned}\mathbf{\Delta}(\Lambda, P) = \{Q \in \mathcal{S}^d_{\geq 0} \quad &: \quad \langle Q, N_{V,\boldsymbol{w}} \rangle > 0 \text{ for all } \boldsymbol{w} \in \Lambda \setminus \operatorname{vert} P \\ &\text{and} \quad \langle Q, N_{V,\boldsymbol{w}} \rangle = 0 \text{ for all } \boldsymbol{w} \in \operatorname{vert} P\}.\end{aligned}$$

Hence $\mathbf{\Delta}(\Lambda, P)$ is in particular a convex cone in $\mathcal{S}^d_{>0}$.

If $\mathbf{\Delta}(\Lambda, P)$ is a ray, $P$ is called an *extreme Delone polytope*, (or *supertope* or *perfect Delone polytope*). Thus up to scaling, they define a unique PQF, respectively up to similarities they define a unique lattice. Examples are the *Schläfli polytope* in dimension 6 which determines the *root lattice* $\mathsf{E}_6$, and the *Gosset polytope* which determines the *root lattice* $\mathsf{E}_7$ in dimension 7. It is known that the Schläfli polytope is the only extreme Delone polytope in dimension 6 (see [**86**]) and that 6 is the lowest dimension $d \geq 2$ in which an extreme Delone polytope exists (see [**104**]). We think it would be very interesting to further study extreme Delone polytopes for various reasons. For example they are connected to local lattice covering maxima, as we shall see in Section 5.6. We refer to [**104**], [**83**] and [**118**] for further reading. Recently, Dutour and Rybnikov [**90**] have developed a new algorithm for the classification of extreme Delone polytopes. With it or variants, it may be possible to classify extreme Delone polytopes in dimension 7 and 8 (cf. [**87**]).

Instead of considering the secondary cone of a single Delone polytope, we may more general consider the *secondary cone* (realization space) of a *polyhedral complex* (union of polyhedra) $\mathcal{C} = \{P_1, P_2, \ldots\}$ with all polyhedra $P_i$ having vertex-sets in $\Lambda$. We define

$$\mathbf{\Delta}(\Lambda, \mathcal{C}) = \{Q \in \mathcal{S}^d_{\geq 0} : \mathcal{C} \subseteq \operatorname{Del}_\Lambda(Q)\}.$$

Then

$$\mathbf{\Delta}(\Lambda, \mathcal{C}) = \bigcap_{P \in \mathcal{C}} \mathbf{\Delta}(\Lambda, P) \qquad \text{and} \qquad \mathbf{\Delta}(\Lambda, P) = \bigcup_{P \in \mathcal{C}} \mathbf{\Delta}(\Lambda, \mathcal{C}),$$

where in the second equation, we can restrict the union to run over all complete Delone subdivisions containing $P$.

Note that our Theorem 4.7 deals with the case of $\mathcal{C}$ being a complete Delone subdivision. For a periodic set $\Lambda$, it shows that $\mathbf{\Delta}(\Lambda, \mathcal{C})$ is a polyhedral cone. For the vertex-set $\mathbb{Z}^d$, Ryshkov and Baranovskii [**216**] consider the case where $\mathcal{C}$ consists of edges (1-faces) only. They call such a complex $\mathcal{C}$ a *$C$-type* and their secondary cone is referred to as a *$C$-type domain.* They show that these $C$-type domains are (relative open) polyhedral cones in $\mathcal{S}^d_{\geq 0}$. Moreover, up to $\mathsf{GL}_d(\mathbb{Z})$-equivalence there exist only finitely many $C$-types. They classify all of the $C$-types with respect to $\mathbb{Z}^d$ up to dimension 5 and based on them solve the lattice covering problem. We are not aware of further, general results on polyhedrality of secondary cones of other polyhedral complexes. In case of a $d$-dimensional polytope $P$, an answer is provided by Proposition 4.22 below.

**4.4.2. Baranovskii cones.** For the special case of a full dimensional simplex $L$ and a lattice $\Lambda$ as vertex-set, the polyhedrality of $\mathbf{\Delta}(\Lambda, L)$ has been answered positively by Baranovskii in [**15**]. His argument works for periodic vertex-sets $\Lambda$ as well. We think it is justified to baptize the secondary cone $\mathbf{\Delta}(\Lambda, L)$ the *Baranovskii cone* of $L$ with respect to $\Lambda$. We note that there is a close connection of the Baranovskii cone of a *unimodular simplex* of $\mathbb{Z}^d$ and the so-called *hypermetric cone* (see [**83**] and [**89**]). A unimodular simplex of a lattice $\Lambda$ is one whose vertices are an affine basis of $\Lambda$.

For every Delone polytope $P$ with vertices in $\Lambda$ and every $d$-dimensional simplex $L$ with $\operatorname{vert} L \subseteq \operatorname{vert} P$, the secondary cone $\mathbf{\Delta}(\Lambda, P)$ is the relative interior of some face of $\overline{\mathbf{\Delta}(\Lambda, L)}$. This follows easily from Proposition 4.8 and shows in particular the following:

PROPOSITION 4.22. *Let $\Lambda \subset \mathbb{R}^d$ be a periodic vertex-set and let $P$ be a polytope with vertices in $\Lambda$. Then $\mathbf{\Delta}(\Lambda, P) \subset \mathcal{S}^d_{>0}$ is a relative open polyhedral cone.*

Due to the polyhedrality of the Baranovskii cone, we can use Algorithm 7 below to decide whether or not a given $d$-dimensional simplex $L$ with vertices in a periodic vertex-set $\Lambda \subset \mathbb{R}^d$ is a Delone simplex for some $Q \in \mathcal{S}^d_{>0}$.

**Input:** An affinely indep. set $V = \{\boldsymbol{v}_0, \dots, \boldsymbol{v}_d\}$ with $\boldsymbol{v}_i \in \Lambda$ for $i = 0, \dots, d$
**Output:** "yes" if $\operatorname{conv} V$ is Delone simplex with respect to $\Lambda$, "no" otherwise
$\mathcal{N} \leftarrow \emptyset$
choose initial $Q \in \mathcal{S}^d_{>0}$
**do**
  $\boldsymbol{c}_Q \leftarrow$ center in $\mathbb{R}^d$ of $\operatorname{conv} V$ with respect to $Q$
  $S \leftarrow$ CVP solutions for $c_Q$ with respect to $\Lambda$ and $Q$
  **if** $V \subseteq S$ **then**
    create $N_{V,\boldsymbol{w}}$ for all $\boldsymbol{w} \in S \setminus V$
    **if** $\{Q' \in \mathcal{S}^d : \langle N_{V,\boldsymbol{w}}, Q' \rangle > 0$ for all $\boldsymbol{w} \in S \setminus V\} \neq \emptyset$ **then**
      **return** "yes"
    **else**
      **return** "no"
    **end if**
  **else**
    choose $\boldsymbol{w} \in S$ (by construction $\notin V$)
    $\mathcal{N} \leftarrow \mathcal{N} \cup \{N_{V,\boldsymbol{w}}\}$
    $\mathcal{C} \leftarrow \{Q' \in \mathcal{S}^d : \langle N, Q' \rangle \geq 0, N \in \mathcal{N}\}$
    **if** $\mathcal{C} = \{0\}$ **then**
      **return** "no"
    **else**
      $Q \leftarrow$ projection (nearest point map) of $Q$ onto $\mathcal{C}$
    **end if**
  **end if**
**end do**

ALGORITHM 7. Deciding if a simplex is Delone for some PQF or not.

Since the algorithm is not used in the remaining, let us explain it only briefly: Just recall the definition of $N_{V,\boldsymbol{w}}$ from (4.3) for a set of affinely independent points $V = \{\boldsymbol{v}_0, \dots, \boldsymbol{v}_d\}$ and $\boldsymbol{w} \in \mathbb{R}^d$. For $V$ there exists a unique center $\boldsymbol{c}_Q$ (which can be found by solving a set of linear equations) and a radius $r > 0$ such that $Q[\boldsymbol{c}_Q - \boldsymbol{v}] = r^2$ for all $\boldsymbol{v} \in V$. By Proposition 4.8 we have

$$Q[\boldsymbol{w} - \boldsymbol{c}_Q] - r^2 = \langle Q, N_{V,\boldsymbol{w}} \rangle. \tag{4.11}$$

By solving a closest vector problem (CVP) for $\boldsymbol{c}_Q$ with respect to $\Lambda$ and $Q$, that is, by determining the set of all vectors $\boldsymbol{v} \in \Lambda$ attaining $\min_{\boldsymbol{v} \in \Lambda} Q[\boldsymbol{c}_Q - \boldsymbol{v}]$, we can test whether or not $V$ gives the vertices of a Delone simplex with respect to $\Lambda$ and $Q$. If this is not the case, we find some $\boldsymbol{w}$ which is closer to $\boldsymbol{c}_Q$ than any $\boldsymbol{v} \in V$. In this case we choose a new PQF $Q'$ such that $Q'[\boldsymbol{w} - \boldsymbol{c}_{Q'}] \geq Q'[\boldsymbol{v} - \boldsymbol{c}_{Q'}]$ for all $\boldsymbol{v} \in V$. Because of (4.11) this can be achieved by projecting $Q$ onto the polyhedral cone, which is defined by the linear inequalities of type $\langle Q, N_{V,\boldsymbol{w}} \rangle \geq 0$.

**4.4.3. Delone simplices.** The most fundamental building blocks of Delone subdivisions are Delone simplices. Only their secondary cones are full dimensional. Therefore, in the classical case with a lattice $\Lambda \subset \mathbb{R}^d$ as a vertex-set, the question arose what kind of Delone simplices do there exist.

All unimodular simplices are *unimodular equivalent*, that is they are equivalent up $\mathsf{GL}_d(\mathbb{Z})$ mappings (lattice base changes) and lattice invariant translations. In particular, unimodular equivalent simplices have the same volume. The *relative volume* (or *normalized volume*) of a lattice simplex $L$ is the volume of $L$ divided by that of a unimodular simplex of $\Lambda$, so that the relative volume equals $\operatorname{vol}(L) \cdot d!$. Equivalently, it equals the index in $\Lambda$ of the sublattice affinely spanned by the vertices of $L$.

Clearly, the relative volume is an invariant with respect to unimodular transformations. Hence, in order to classify possible Delone simplices up to unimodular equivalence, a first question, already raised by Delone in [**76**], is what is the maximum relative volume $\operatorname{mv}(d)$ of $d$-dimensional Delone simplices. The sequence $\operatorname{mv}(d)$ is (weakly) increasing, since from a Delone simplex for a lattice $\Lambda \subset \mathbb{R}^d$ one can easily construct another of the same relative volume for the lattice $\Lambda \times \mathbb{Z} \subset \mathbb{R}^{d+1}$.

To see that the relative volume is bounded from above, we may use the following argument, which is Lemma 14.2.5 in [**83**] (attributed to L. Lovász): Given a Delone simplex $L$ of some PQF, the volume of the centrally symmetric difference body $L - L = \{\boldsymbol{v} - \boldsymbol{v}' : \boldsymbol{v}, \boldsymbol{v}' \in L\}$ is $\operatorname{vol}(L - L) = \binom{2d}{d} \operatorname{vol}(L)$ (see [**207**]). The polytope $L - L$ does not contain elements of $L \setminus \{0\}$ in its interior. Thus, by Minkowski's convex body theorem 1.3 we know that $\operatorname{vol}(L - L) \leq 2^d$. Putting things together we get

$$\operatorname{mv}(d) \leq \frac{2^d d!}{\binom{2d}{d}} \sim \sqrt{2}\pi d \left(\frac{d}{2e}\right)^d . \tag{4.12}$$

As a consequence, we see that there exist only finitely many unimodular inequivalent Delone simplices with respect to $\Lambda$. So naturally, the question to classify all of them arises. Voronoi knew that up to dimension $d = 4$ all Delone simplices have relative volume 1, while there are Delone simplices of volume 2 for $d = 5$. In [**16**] Baranovskii proved $\operatorname{mv}(5) = 2$, and later Baranovskii and Ryshkov [**23**] proved $\operatorname{mv}(6) = 3$. Dutour classified all 6-dimensional Delone polytopes in [**86**]. We think, for example based on Algorithm 7, it is possible to classify Delone simplices in dimension 7 and maybe even 8.

**4.4.4. Delone simplices of large relative volume.** Ryshkov [**213**] was the first who proved that relative volumes of Delone simplices are not bounded when the dimension goes to infinity. More precisely, for every $k \in \mathbb{N}$ he constructed Delone simplices of relative volume $k$ in dimension $2k+1$, establishing that $\operatorname{mv}(d) \geq \left\lfloor \frac{d-1}{2} \right\rfloor$. Ryshkov's constructive proof (which is already in Ryshkov's dissertation, according to Baranovskii [**16**]) is based on the *Coxeter lattices* $A_d^k$ (see Section 5.3.6

for a definition), which Coxeter [**72**] already suspected to contain such large Delone simplices (cf. [**105**]).

Recently, the lower bound of Ryshkov was improved to the still linear lower bound $\mathrm{mv}(d) \geq d-3$ by Erdahl and Rybnikov [**105**]. They show that for $d \geq 4$, the simplices

$$L = \mathrm{conv}\{\mathbf{0}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_{d-1}}, k\boldsymbol{e_d} + \sum_{i=1}^{d-1} \boldsymbol{e_i}\}$$

are Delone simplices (with respect to the vertex-set $\mathbb{Z}^d$ and a suitable PQF) if $1 \leq k \leq d-3$.

In this section we prove the following two lower bounds on $\mathrm{mv}(d)$:

THEOREM 4.23. *For every pair of dimensions $d_1$ and $d_2$,*

$$\mathrm{mv}(d_1 + d_2) \geq \mathrm{mv}(d_1)\,\mathrm{mv}(d_2).$$

THEOREM 4.24. *For every dimension of the form $d = 2^n - 1$,*

$$\mathrm{mv}(d) \geq (d+1)^{(d+3)/2}/4^d.$$

Theorem 4.23 immediately implies exponential lower bounds on $\mathrm{mv}(d)$. For example, $\mathrm{mv}(5) = 2$ gives $\mathrm{mv}(d) \geq 2^{\lfloor d/5 \rfloor} \sim 1.1487^d$. Even better, it is known that $\mathrm{mv}(24) \geq 20480$. Indeed, the Delone subdivision of the Leech lattice, which was determined by Borcherds, Conway, Queen, Parker and Sloane [**69**, Chapter 25], contains simplices of relative volume 20480 (the simplex denoted $a_1^{24}a_1$ in their classification). Therefore we obtain:

COROLLARY 4.25.

$$\mathrm{mv}(d) \geq 20480^{\lfloor d/24 \rfloor} \sim 1.5123^d.$$

Theorem 4.24 gives a much better lower bound asymptotically:

COROLLARY 4.26.

$$\log(\mathrm{mv}(d)) \in \Theta(d \log d).$$

PROOF. For the upper bound we use (4.12). For the lower bound, let $2^n$ be the largest power of two that is smaller or equal to $d+1$ (so that $2^n \geq (d+1)/2$). Theorem 4.24, together with the monotonicity of $\mathrm{mv}(d)$, gives:

$$\mathrm{mv}(d) \geq \mathrm{mv}(2^n - 1) \geq \frac{(2^n)^{(2^n+2)/2}}{4^{2^n-1}} \in 2^{\Theta(n2^n)} = 2^{\Theta(d \log d)}.$$

□

More precisely, the arguments in this proof say that

$$\frac{1}{4} \leq \liminf \frac{\log \mathrm{mv}(d)}{d \log d} \leq \limsup \frac{\log \mathrm{mv}(d)}{d \log d} \leq 1.$$

Similarly, Theorem 4.24 directly implies

$$\frac{1}{2} \leq \limsup \frac{\log \mathrm{mv}(d)}{d \log d}.$$

We do not know whether $\lim \frac{\log \mathrm{mv}(d)}{d \log d}$ exists.

Note that Theorem 4.23 and Theorem 4.24 together can be used to construct Delone simplices with large relative volume in every dimension. For example, by the monotonicity of $\mathrm{mv}(d)$ we find simplices with large relative volume for every power of 2. Representing $d$ in binary we may use Theorem 4.24 to obtain Delone simplices with large relative volumes for every power-of-two summand. Application of Theorem 4.23 then yields a $d$-dimensional Delone simplex whose relative volume is the product ot these relative volumes. Note however, for $d = 24$ this construction gives a simplex of relative volume 128, which is far from the relative volume 20480 of the Delone simplex of the Leech lattice mentioned above.

4.4.4.1. *Proof of Theorem 4.23.* Theorem 4.23 follows from the fact that an orthogonal product of simplices decomposes into simplices with relative volume being the product of the individual relative volumes. Let us be more precise: Let $L_1 = \operatorname{conv}\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_{d_1}\} \subseteq \mathbb{R}^{d_1}$ be a Delone simplex of the lattice $\Lambda_1$ with relative volume $\operatorname{mv}(d_1)$, and let $L_2 = \operatorname{conv}\{\boldsymbol{w}_0, \dots, \boldsymbol{w}_{d_2}\} \subseteq \mathbb{R}^{d_2}$ be a Delone simplex of the lattice $\Lambda_2$ with relative volume $\operatorname{mv}(d_2)$. Then, the direct product

$$L_1 \times L_2 = \operatorname{conv}\{(\boldsymbol{v}_i, \boldsymbol{w}_j) : i = 0, \dots, d_1,\ j = 0, \dots, d_2\} \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$$

is a $(d_1 + d_2)$-dimensional Delone polytope of the lattice $\Lambda = \Lambda_1 \times \Lambda_2$. Let $\Lambda_1'$, $\Lambda_2'$ and $\Lambda' = \Lambda_1' \times \Lambda_2'$ denote the lattices affinely generated by the vertices of $L_1$, $L_2$ and $L_1 \times L_2$, respectively. By the classical theory of Voronoi (see Section 4.1) we know that by a suitable infinitesimal change of the PQF $Q$ that induces $\Lambda$, the Delone polytope $L_1 \times L_2$ is triangulated into Delone simplices, which hence have vertices in the sublattice $\Lambda'$ (more precisely, in a perturbation of it). Since the index of $\Lambda'$ in $\Lambda$ is precisely $\operatorname{mv}(d_1)\operatorname{mv}(d_2)$, these Delone simplices have relative volume at least that number.

Remark 4.27. The product of two simplices $L_1$ and $L_2$ is a *totally unimodular* polytope, meaning that all the simplices spanned by a subset of its vertices have the same volume. In particular, the Delone simplices that we obtain in the last step of the proof have relative volume *exactly* $\operatorname{mv}(d_1)\operatorname{mv}(d_2)$ (cf., for example, [**128**]). As a consequence, every triangulation of $L_1 \times L_2$ consists of exactly $\binom{d_1+d_2}{d_1}$ simplices.

4.4.4.2. *Proof of Theorem 4.24.* Remember that a Hadamard matrix of order $d$ is a $d \times d$ matrix with elements $+1$ and $-1$ in which distinct columns are orthogonal. Hadamard matrices exist at least whenever $d$ is a power of two and conjecturally whenever $d$ is a multiple of four (cf. [**69**, Chapter 3,2.13]; for a recent survey on Hadamard matrices see [**100**]).

By multiplying columns with $\pm 1$ we normalize columns of a Hadamard matrix $H$ so that the entries of the first row are all $+1$. Then, from $H$ we get the $(d-1) \times d$ matrix $\tilde{H}$ with elements 0 and 1 by deleting the first row and replacing $+1$ by 0, and $-1$ by 1. The columns of $\tilde{H}$ form the vertex-set of a regular $(d-1)$ simplex with edge length $\sqrt{d/2}$ and volume $(\det H)/2^{d-1} = d^{d/2}/2^{d-1}$. This is the maximum volume of a simplex contained in the $(d-1)$ unit cube $[0,1]^{d-1}$. Such a simplex is called a $(d-1)$ Hadamard simplex.

A standard construction for Hadamard matrices of order $2^n$ is as follows

$$H_1 = (1), \quad H_{2^n} = \begin{pmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ -H_{2^{n-1}} & H_{2^{n-1}} \end{pmatrix}.$$

Let $d = 2^n - 1$ and consider the matrix $\tilde{H}_{2^n}$ associated to the particular Hadamard matrix $H_{2^n}$ constructed this way. It is well known, and easy to see, that the columns of $\tilde{H}_{2^n}$ form an $n$-dimensional subspace of $\mathbb{F}_2^d$. This subspace is also known as the $[d, n]$ binary simplex code or the dual of the $[d, d-n]$ binary Hamming code.

Linearity implies that the following is a $d$-dimensional sublattice of $\mathbb{Z}^d$:

$$\Lambda(\tilde{H}_{2^n}) = \{(v_1, \dots, v_d) \in \mathbb{Z}^d : (v_1 \bmod 2, \dots, v_d \bmod 2) \in \tilde{H}_{2^n}\}.$$

This procedure is "Construction A" in [**69**, Chapters 5 and 7]. The lattice $\Lambda(\tilde{H}_{2^n})$ has determinant $2^{d-n}$ in $\mathbb{Z}^d$, since it contains $2^n$ of the $2^d$ vertices in every lattice unit cube. Moreover, the Hadamard simplex defined by $\tilde{H}_{2^n}$ is a Delone simplex in $\Lambda(\tilde{H}_{2^n})$ because the sphere around the unit cube does not contain more lattice points than the columns of $\tilde{H}_{2^n}$. The relative volume of the Hadamard simplex is

$$\frac{(d+1)^{(d+1)/2}/2^d}{2^{d-n}} = \frac{(d+1)^{(d+3)/2}}{4^d}.$$

4.4.4.3. *Remarks on related problems.* The determination of $\mathrm{mv}(d)$ is clearly related (but not equivalent) to that of what is the minimal number of translational orbits of Delone simplices in Delone triangulations of $d$-dimensional lattices. Let us denote this number $\mathrm{dt}(d)$. Since the sum of volumes of representatives from each orbit must equal $d!$, the upper bound for $\mathrm{mv}(d)$ implies

$$\mathrm{dt}(d) \geq \binom{2d}{d} 2^{-d} = \Omega(2^d/\sqrt{d}).$$

For an upper bound, we can use essentially the same trick as in the proof of Theorem 4.23 to obtain:

PROPOSITION 4.28. *There is a constant $c < 1$ such that $\mathrm{dt}(d) \leq c^d d!$, for every $d \geq 5$.*

PROOF. Start with Delone triangulations with the minimal number of translational orbits $\mathrm{dt}(d_1)$ and $\mathrm{dt}(d_2)$ in dimensions $d_1$ and $d_2$, and consider the product. As before, a perturbation of the product lattice produces a Delone triangulation with $\binom{d_1+d_2}{d_1} \mathrm{dt}(d_1)\,\mathrm{dt}(d_2)$ translational orbits, from which we deduce

$$\frac{\mathrm{dt}(d_1+d_2)}{(d_1+d_2)!} \leq \frac{\mathrm{dt}(d_1)}{d_1!} \cdot \frac{\mathrm{dt}(d_2)}{d_2!}.$$

Hence,

$$\left(\frac{\mathrm{dt}(d_1+d_2)}{(d_1+d_2)!}\right)^{\frac{1}{d_1+d_2}} \leq \max\left\{\left(\frac{\mathrm{dt}(d_1)}{d_1!}\right)^{\frac{1}{d_1}}, \left(\frac{\mathrm{dt}(d_2)}{d_2!}\right)^{\frac{1}{d_2}}\right\}.$$

Since in every dimension $d \geq 5$ there are Delone simplices of volume greater than one, we have constants $c_d < 1$ such that $\mathrm{dt}(d) = {c_d}^d d!$. It suffices now to take as the global constant $c$ for the statement the minimum of $c_5$, $c_6$, $c_7$, $c_8$ and $c_9$. $\square$

Of course, in order to get good (asymptotic) values of the constant in the statement, one needs to compute the number of translational orbits of Delone simplices in "good lattices".

A similar problem that has attracted attention both for theoretical and applied reasons is the determination of the minimum number of simplices in a triangulation of the $d$-dimensional cube $[0,1]^d$. In fact, our proof of Theorem 4.23, as well as the upper bound in the previous remark, is essentially an adaptation of the technique used by Haiman [**128**] to construct "simple and relatively efficient triangulations of the $n$-cube".

The situation for this problem is that the best lower bound known for the minimum size $\mathrm{t}(d)$ of a triangulation of $[0,1]^d$ is

$$\sqrt{\frac{6}{d+1}} \sim \frac{\sqrt{6}}{\sqrt[d]{2}(d+1)^{\frac{d+1}{2d}}} \leq \sqrt[d]{\frac{\mathrm{t}(d)}{d!}},$$

obtained by Smith [**240**]. The best (asymptotic) upper bound is

$$\lim_{d\to\infty} \sqrt[d]{\frac{\mathrm{t}(d)}{d!}} \leq 0.816,$$

due to Orden and Santos [**194**]. For more information on this topic see [**273**].

CHAPTER 5

# Local Analysis of Coverings and Applications

In this chapter we consider the sphere covering and sphere packing-covering problem, in particular for lattices. In Section 5.1 we formulate algorithms that allow to solve the two lattice restricted problems (in principle) by solving a finite number of convex optimization problems. The presented algorithms depend crucially on the reduction theories developed in Chapter 4. In Section 5.2 we provide tools for a local analysis, allowing us to obtain computationally, mathematically rigorous proofs. We give local lower bounds which enable us to prove local optimality for certain lattices. In Section 5.3 we present new lattice coverings and packing-coverings, which we found by application of our methods. In Section 5.4 we apply our algorithms to a problem in algebraic number theory, the classification of all totally real, thin algebraic number fields. In Section 5.5 we apply our techniques to show that the Leech lattice gives a local optimum among lattices. The same method does not work for the $\mathsf{E}_8$ root lattice and in fact, a local analysis reveals that it is not locally optimal. Even worse, it is "almost" a local maximum. Such *"covering pessima"*, as well as "real" local covering maxima are considered in Section 5.6. We in particular show that the only local lattice covering maximum in dimensions $d = 2, \ldots, 6$ is the root lattice $\mathsf{E}_6$. In Section 5.7 we give a connection of local covering maxima within a linear subspace $T \subseteq \mathcal{S}^d$ to a famous conjecture of Minkowski. We obtain an algorithm which can be used to prove Minkowski's conjecture in a given dimension.

## 5.1. Optimizing sphere coverings

**5.1.1. Coverings and PQFs.** As in case of the lattice sphere packing problem, it is convenient to work in the space of positive definite quadratic forms. The *inhomogeneous minimum* $\mu(Q)$ of a $Q \in \mathcal{S}^d_{>0}$ is given by

$$\mu(Q) = \max_{\boldsymbol{x}\in\mathbb{R}^d} \min_{\boldsymbol{v}\in\mathbb{Z}^d} Q[\boldsymbol{x} - \boldsymbol{v}]. \tag{5.1}$$

The inhomogeneous minimum and and its generalizations have been extensively studied in the geometry of numbers (cf. for example [**124**]). For a corresponding lattice $L = A\mathbb{Z}^d$, with $A \in \mathsf{GL}_d(\mathbb{R})$ such that $Q = A^tA$, we have

$$\mu(L) = \sqrt{\mu(Q)}.$$

Thus solving the lattice covering problem in $\mathbb{R}^d$ means to minimize

$$\Theta(Q) = \sqrt{\frac{\mu(Q)^d}{\det Q}} \cdot \operatorname{vol} B^d = \Theta(L)$$

on $\mathcal{S}^d_{>0}$. Scaling of $Q$ by a positive real number leaves $\Theta$ invariant. Consequently, we can restrict our attention to those PQFs $Q$ with $\mu(Q) = \beta > 0$ say. Hence we solve the lattice covering problem if we solve the following optimization problem:

$$\tag{5.2} \boxed{\begin{array}{ll} \textbf{maximize} & \det Q \\ \textbf{subject to} & Q \text{ is a positive definite matrix,} \\ & \mu(Q) = \beta, \end{array}}$$

where the optimization variables $q_{ij}$ are the entries of the PQF $Q$. The major disadvantage of this optimization problem is that the second constraint is not expressible as a convex condition in the variables $q_{ij}$ and that the problem has many local maxima. A PQF $Q$ giving a locally optimal solution of the above optimization problem yields a *locally optimal lattice covering.*

Similarly we see that solving the lattice packing-covering problem in $\mathbb{R}^d$ means to minimize

$$\gamma(Q) = 2\sqrt{\frac{\mu(Q)}{\lambda(Q)}} = \gamma(L)$$

among all PQFs $Q \in \mathcal{S}^d_{>0}$. Since $\gamma$ is homogeneous we can again assume $\mu(Q) = \beta > 0$, and the lattice packing-covering problem is equivalent to the following optimization problem.

$$\tag{5.3} \boxed{\begin{array}{ll} \textbf{maximize} & \lambda(Q) \\ \textbf{subject to} & Q \text{ is a positive definite matrix,} \\ & \mu(Q) = \beta. \end{array}}$$

A locally optimal solution of this optimization problem yields a *locally optimal lattice packing-covering.*

In the following sections we show how these two optimization problems can be solved by breaking them up into a finite number of special convex optimization problems. Using the theory developed in Chapter 4 we can generalize this to the situation, where $\mathbb{Z}^d$ is replaced by some rational standard periodic vertex-set. For this we extend the definition of the inhomogeneous minimum to an arbitrary discrete set $\Lambda \subset \mathbb{R}^d$. We define the *inhomogeneous minimum of a PQF $Q$ with respect to* $\Lambda$ by

$$\tag{5.4} \mu_\Lambda(Q) = \sup_{\boldsymbol{x} \in \mathbb{R}^d} \inf_{\boldsymbol{v} \in \Lambda} Q[\boldsymbol{x} - \boldsymbol{v}].$$

We can obtain the best sphere covering density (the best packing-covering constant) among sets $A\Lambda$ with $A \in \mathsf{GL}_d(\mathbb{R})$ by maximizing $\det Q$ (maximizing $\lambda(Q)$) on the set of all PQFs $Q$ with $\mu_\Lambda(Q) = 1$. This maximum may not exist. However, if $\Lambda$ is a rational standard periodic set, then it does exist and, as we shall see, we can determine it algorithmically.

**5.1.2. Convex optimization with LMI constraints.** In this section we introduce determinant maximization problems, which are convex programming problems with linear matrix inequality constraints. In a sense they are equivalent to the better known semidefinite programming problems. For both classes efficient algorithms and implementations are available. In Section 5.1.4 we will see how the lattice covering problem and the lattice packing-covering problem can naturally be formulated as a finite number of determinant maximization problems.

Following Vandenberghe, Boyd, and Wu [**258**] we say that a *determinant maximization problem* is an optimization problem of the form

$$\text{(5.5)} \qquad \boxed{\begin{array}{ll} \textbf{minimize} & \boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x}) \\ \textbf{subject to} & G(\boldsymbol{x}) \text{ is a positive definite matrix,} \\ & F(\boldsymbol{x}) \text{ is a positive semidefinite matrix.} \end{array}}$$

The optimization vector is $\boldsymbol{x} \in \mathbb{R}^d$, the objective vector is $\boldsymbol{c} \in \mathbb{R}^d$, and $G : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^d \to \mathbb{R}^{n\times n}$ are affine maps:

$$\begin{array}{rcl} G(\boldsymbol{x}) & = & G_0 + x_1G_1 + \cdots + x_dG_d, \\ F(\boldsymbol{x}) & = & F_0 + x_1F_1 + \cdots + x_dF_d, \end{array}$$

where $G_i \in \mathbb{R}^{m\times m}$ and $F_i \in \mathbb{R}^{n\times n}$, for $i = 0, \ldots, d$, are symmetric matrices. In the sequel we will write $G(\boldsymbol{x}) \succ 0$ and $F(\boldsymbol{x}) \succeq 0$ for the *linear matrix inequalities* (LMIs) defining the constraints of the determinant maximization problem. As a special case, our formulation reduces to a semidefinite programming problem whenever $G(\boldsymbol{x})$ is the identity matrix for all $\boldsymbol{x} \in \mathbb{R}^d$.

Nesterov and Nemirovskii [**190**] developed a framework for the design of efficient interior-point algorithms for general and specific classes of convex programming problems. There (§6.4.3), they also showed that the determinant maximization problem can be transformed into a semidefinite programming problem by a transformation which can be computed in polynomial time. Nevertheless it is faster to solve the determinant maximization problem directly. Vandenberghe, Boyd, Wu [**258**] and independently Toh [**252**] gave interior-point algorithms for the determinant maximization problem. Both algorithms fit into the general framework of Nesterov and Nemirovskii. For our implementation we use the software package `MAXDET` [**280**] of Wu, Vandenberghe, and Boyd as a subroutine.

One nice feature of determinant maximization problems is that there is a duality theory similar to the one of linear programming, which allows one to compute *certificates* for a range in which the optimum is attained, the so-called *duality gap*. The dual problem of problem (5.5) is (see [**258**, Sec. 3])

$$\text{(5.6)} \qquad \boxed{\begin{array}{ll} \textbf{maximize} & \log\det W - \operatorname{trace}(G_0W) - \operatorname{trace}(F_0Z) + m \\ \textbf{subject to} & \operatorname{trace}(G_iW) + \operatorname{trace}(F_iZ) = \boldsymbol{c}_i \text{ for } i = 1, \ldots, d, \\ & W \succ 0,\ Z \succeq 0. \end{array}}$$

Here, $W \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{n\times n}$ are symmetric matrices of the same size as the $G_i$ and $F_i$ respectively.

The knowledge of a vector $\boldsymbol{x}$ with $G(\boldsymbol{x}) \succ 0$ and $F(\boldsymbol{x}) \succeq 0$, and of a pair $(W, Z)$, with $\operatorname{trace}(G_iW) + \operatorname{trace}(F_iZ) = \boldsymbol{c}_i$ for $i = 1, \ldots, d$, and $W \succ 0$, and $Z \succeq 0$, gives upper and lower bounds for the optimal value $p^*$ of (5.5) by

$$\log\det W - \operatorname{trace}(G_0W) - \operatorname{trace}(F_0Z) + m \leq p^* \leq \boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x}).$$

If $\boldsymbol{x}$, $W$ and $Z$ have rational entries only, we can compute lower and upper bounds of $p^*$ that are mathematical rigorous. For the proofs in Section 5.3 we have therefore developed a program `rmd` (*rigorous MAXDET*) which calls `MAXDET` to find floating point values of a vector $\boldsymbol{x}$ and a pair $(W, Z)$ satisfying the constraints. Then it approximates $\boldsymbol{x}$ and $(W, Z)$ by rational numbers and checks whether these rational approximations satisfy the constraints. In such a case they guarantee a certified duality gap. In case of a certified zero duality gap, we have a *certificate* (computational proof) for local optimality. The interested reader can download `rmd` [**285**].

**5.1.3. An LMI constraint for the inhomogeneous minimum.** In this section we will give a linear matrix inequality in the parameters $q_{ij}$ of a PQF $Q = (q_{ij})$, which is satisfied if and only if the inhomogeneous minimum of $Q$ is bounded, say $\mu(Q) \leq \beta$ for some $\beta > 0$. For this it is crucial to observe that $\boldsymbol{x}^t Q \boldsymbol{y} = \langle Q, \boldsymbol{x}\boldsymbol{y}^t \rangle$ is a linear expression in the parameters $(q_{ij})$. The PQF $Q = (q_{ij})$ defines the inner product of a Euclidean space $(\mathbb{R}^d, (\cdot,\cdot))$ by $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^t Q \boldsymbol{y}$. It induces a *distance*

$$|\boldsymbol{x} - \boldsymbol{y}| = (\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})^{\frac{1}{2}}.$$

From a result of Delone, Dolbilin, Ryshkov and Stogrin [**77**, §3] we can extract the following proposition which will be central in our further discussion. In it we denote by $\mu_L(Q)$ the *squared circumradius* of $L$ with respect to $(\cdot,\cdot)$ (respectively $Q$).

Proposition 5.1. *Let $L = \operatorname{conv}\{\boldsymbol{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\} \subseteq \mathbb{R}^d$ be a $d$-dimensional simplex. Then $\mu_L(Q) \leq \beta$ for some $\beta > 0$ if and only if the following linear matrix inequality (in the parameters $q_{ij}$) is satisfied:*

$$\mathrm{BR}_{L,\beta}(Q) = \begin{pmatrix} 4\beta & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_2,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_1,\boldsymbol{v}_d) \\ (\boldsymbol{v}_2,\boldsymbol{v}_2) & (\boldsymbol{v}_2,\boldsymbol{v}_1) & (\boldsymbol{v}_2,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_2,\boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d,\boldsymbol{v}_d) & (\boldsymbol{v}_d,\boldsymbol{v}_1) & (\boldsymbol{v}_d,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \end{pmatrix} \succeq 0.$$

In [**77**] Delone et al. used this proposition as the key ingredient for showing that the set of PQFs which determine a circumsphere of $L$ with radius at most $R$ is convex and bounded (see Proposition 5.6). Because of the importance of Proposition 5.1 and for the convenience of the reader we give a short proof here.

Proof. We make use of Cayley-Menger determinants. The Cayley-Menger determinant of $d+1$ points $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d$ with pairwise distances $|\boldsymbol{v}_i - \boldsymbol{v}_j|$ is

$$\mathrm{CM}(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d) = \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & |\boldsymbol{v}_0 - \boldsymbol{v}_0|^2 & \ldots & |\boldsymbol{v}_0 - \boldsymbol{v}_d|^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & |\boldsymbol{v}_d - \boldsymbol{v}_0|^2 & \ldots & |\boldsymbol{v}_d - \boldsymbol{v}_d|^2 \end{vmatrix}.$$

The squared circumradius $R^2$ of the simplex $L$ equals (cf. [**38**, Prop. 9.7.3.7])

$$R^2 = -\frac{1}{2} \cdot \frac{\det\left(|\boldsymbol{v}_i - \boldsymbol{v}_j|^2\right)_{0 \leq i,j \leq d}}{\mathrm{CM}(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)}.$$

Replacing $|\boldsymbol{x} - \boldsymbol{y}|^2$ by $(\boldsymbol{x},\boldsymbol{x}) - 2(\boldsymbol{x},\boldsymbol{y}) + (\boldsymbol{y},\boldsymbol{y})$, using $\boldsymbol{v}_0 = \boldsymbol{0}$, and performing elementary transformations of the determinants turns the above formula into

$$R^2 = -\frac{1}{4} \cdot \frac{\begin{vmatrix} 0 & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_2,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_1,\boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d,\boldsymbol{v}_d) & (\boldsymbol{v}_d,\boldsymbol{v}_1) & (\boldsymbol{v}_d,\boldsymbol{v}_2) & \ldots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \end{vmatrix}}{\det\left((\boldsymbol{v}_i,\boldsymbol{v}_j)\right)_{1 \leq i,j \leq d}}. \tag{5.7}$$

The inequality $R \leq \beta^{\frac{1}{2}}$ is then equivalent to

$$4\beta \cdot \det\left((\boldsymbol{v}_i, \boldsymbol{v}_j)\right)_{1\leq i,j\leq d} + \begin{vmatrix} 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \geq 0,$$

which is equivalent to

$$\begin{vmatrix} 4\beta & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \geq 0.$$

The minors in the lower right are all determinants of Gram matrices and therefore non-negative. Hence, the matrix $\mathrm{BR}_{L,\beta}(Q)$ is positive semidefinite. $\square$

COROLLARY 5.2. *For any $d$-dimensional simplex $L \subseteq \mathbb{R}^d$ with vertex at $\mathbf{0}$, $\beta > 0$ and $Q \in \mathcal{S}^d_{>0}$ we have*

$$|\mathrm{BR}_{L,\beta}(Q)| \geq 0 \quad \Longleftrightarrow \quad \mathrm{BR}_{L,\beta}(Q) \succeq 0.$$

EXAMPLE 5.3. Let us compute the matrix linear inequality $\mathrm{BR}_{L,\beta}(Q) \succeq 0$, $Q = \left(\begin{smallmatrix} q_{11} & q_{21} \\ q_{21} & q_{22} \end{smallmatrix}\right)$, for the two-dimensional simplex $L = \mathrm{conv}\{\binom{0}{0}, \binom{1}{0}, \binom{1}{1}\}$. We have

$$\begin{aligned} \mathrm{BR}_{L,\beta}(Q) &= \begin{pmatrix} 4\beta & q_{11} & q_{11}+2q_{21}+q_{22} \\ q_{11} & q_{11} & q_{11}+q_{21} \\ q_{11}+2q_{21}+q_{22} & q_{11}+q_{21} & q_{11}+2q_{21}+q_{22} \end{pmatrix} \\ &= \begin{pmatrix} 4\beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + q_{11}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + q_{21}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} + q_{22}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

$\square$

For a Delone polytope $P$ of $Q$ (not necessarily a simplex) let $\mu_P(Q)$ denote its squared circumradius with respect to $Q$. Then $\mu_P(Q) \leq \beta$ if and only if $\mu_L(Q) \leq \beta$ for some $d$-dimensional simplex $L$ with vertices in $\mathrm{vert}\, P$. Two Delone polytopes $P$ and $P'$ with vertices in $\mathbb{Z}^d$ are called $\mathsf{GL}_d(\mathbb{Z})$-*equivalent* with respect to a given $Q \in \mathcal{S}^d_{>0}$, if there exists a $U \in \mathrm{Aut}\, Q$ and a $\boldsymbol{v} \in \mathbb{Z}^d$ with $P = \boldsymbol{v} + UP'$. Note that in particular $P$ and $-P$ are always $\mathsf{GL}_d(\mathbb{Z})$-equivalent.

Since a block matrix is semidefinite if and only if the blocks are semidefinite, we have the following proposition which allows us to express the constraint "$\mu(Q) \leq \beta$" locally by a single linear matrix inequality.

PROPOSITION 5.4. *Let $Q \in \mathcal{S}^d_{>0}$ be a PQF and let $P_1, \dots, P_n$ be a representative system of $d$-dimensional Delone polytopes in $\mathrm{Del}_{\mathbb{Z}^d}(Q)$, which are $\mathsf{GL}_d(\mathbb{Z})$-inequivalent with respect to $Q$. For every $P_i$ choose a $d$-dimensional simplex $L_i$ with $\mathrm{vert}\, L_i \subseteq \mathrm{vert}\, P_i$. Then for $\beta > 0$*

$$\mu(Q) \leq \beta \quad \Longleftrightarrow \quad \begin{pmatrix} \boxed{\mathrm{BR}_{L_1,\beta}(Q)} & 0 & \dots & 0 \\ 0 & \boxed{\mathrm{BR}_{L_2,\beta}(Q)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boxed{\mathrm{BR}_{L_n,\beta}(Q)} \end{pmatrix} \succeq 0.$$

Given a Delone subdivision $\mathcal{D}$ with vertex-set $\mathbb{Z}^d$, the proposition can in particular be applied simultaneously to all PQFs $Q$ in the secondary cone $\mathbf{\Delta}(\mathcal{D})$. If we additionally restrict to PQFs $Q$ which are invariant with respect to some finite subgroup $G \subset \mathsf{GL}_d(\mathbb{Z})$, we may reduce the number of blocks to one for each *$G$-equivalence* class of Delone polytopes.

**5.1.4. Algorithms for solving covering problems.** In this section we describe algorithms which in principal solve the lattice covering problem and the lattice packing-covering problem in any dimension $d$. They are based on Proposition 5.4, which allows to formulate these covering problems locally as determinant maximization problems. The size of these optimization problems can be reduced by Proposition 5.4, if we restrict to PQFs invariant with respect to some finite group $G \subset \mathsf{GL}_d(\mathbb{Z})$. This is the essential ingredient for our systematic search for new best known configurations, which we describe in Section 5.3. We also indicate in this section how this can be generalized to rational standard periodic sets.

Our algorithm for the lattice covering problem computes all locally optimal lattice coverings of a given dimension. These are only finitely many. In fact, by Proposition 5.7, for every fixed Delone triangulation $\mathcal{D}$ there exists at most one PQF lying in the topological closure of the secondary cone of $\mathcal{D}$ giving a locally optimal covering density. So, we fix a Delone triangulation and try to find the PQF which minimizes the density function in the topological closure of the secondary cone of the fixed Delone triangulation. We formulate this restricted lattice covering problem as a determinant maximization problem. Our algorithm for the lattice packing-covering problem operates similarly. For every Delone triangulation we have to solve a semidefinite programming problem.

5.1.4.1. *Solving the Lattice Covering Problem.* For every Delone triangulation $\mathcal{D}$ we solve the optimization problem

$$\boxed{\begin{array}{ll} \textbf{maximize} & \det Q \\ \textbf{subject to} & Q \in \overline{\mathbf{\Delta}(\mathcal{D})}, \\ & \mu(Q) \leq 1. \end{array}} \tag{5.8}$$

The relaxation of no longer requiring $\mu(Q) = 1$ in the third constraint does not give more optimal solutions because if $Q$ satisfies the constraints, then so does $\frac{1}{\mu(Q)}Q$. Now, we have to show that this is indeed a determinant maximization problem. We have seen in Section 4.1 that the first constraint can be expressed with inequalities linear in $q_{ij}$. The constraint $\mu(Q) \leq 1$ can be expressed by a linear matrix inequality as in Proposition 5.4.

The optimization vector $\boldsymbol{x} \in \mathbb{R}^{d(d+1)/2}$ is the vector of coefficients of $Q$. The linear matrix inequality $G(\boldsymbol{x}) \succ 0$ is given by $G(\boldsymbol{x}) = Q$. We encode the two other constraints $Q \in \overline{\mathbf{\Delta}(Q)}$ and $\mu(Q) \leq 1$ by block matrices in the linear matrix inequality $F(\boldsymbol{x}) \succeq 0$. For any linear inequality which is needed to describe the secondary cone we have a $1 \times 1$ block matrix. For any inequivalent $d$-dimensional simplex $L \in \mathcal{D}$ we have the $(d+1) \times (d+1)$ block matrix $\mathrm{BR}_{L,1}(Q)$.

5.1.4.2. *Solving the Lattice Packing-Covering Problem.* Along the same lines as above we formulate the lattice packing-covering problem as a finite number of semidefinite programming problems. From the discussion for the lattice covering problem we know how to deal with the constraint $\mu(Q) = 1$. But how do we maximize $\lambda(Q)$? A theorem of Voronoi (see [**263**, §55] or [**69**, Ch. 21, Th. 10]) implies that a shortest vector $\boldsymbol{v} \in \operatorname{Min} Q$ gives the edge $[\mathbf{0}, \boldsymbol{v}]$ in the Delone subdivision of $Q$. In a fixed Delone subdivision $\mathcal{D}$ there are only finitely many (at most $2(2^d-1)$, see [**263**, §55]) edges of the form $[\mathbf{0}, \boldsymbol{v}]$. We can maximize $\lambda(Q)$ for $Q \in \overline{\mathbf{\Delta}(\mathcal{D})}$ as follows: We introduce a new variable $C$ which we want to maximize subject to the constraints $Q[\boldsymbol{v}] \geq C$ where $\boldsymbol{v}$ runs through all edges of the form $[\mathbf{0}, \boldsymbol{v}]$ in $\mathcal{D}$. This assures $\lambda(Q) = C$ when $C$ attains a maximum. All these expressions are linear in the coefficients $q_{ij}$ of $Q$. Hence we have to solve the following semidefinite programming problem for every inequivalent Delone triangulation $\mathcal{D}$ in order to solve the lattice packing-covering problem:

$$\boxed{\begin{array}{ll} \textbf{maximize} & C \\ \textbf{subject to} & Q \in \overline{\mathbf{\Delta}(\mathcal{D})}, \\ & \mu(Q) \leq 1, \\ & Q[\boldsymbol{v}] \geq C, \text{ where } [\mathbf{0}, \boldsymbol{v}] \text{ is an edge in } \mathcal{D}. \end{array}} \tag{5.9}$$

5.1.4.3. *Extension to periodic sets.* We can adapt the above algorithms for the lattice sphere covering and the lattice packing-covering problem, to fit our extensions of Voronoi's theory to periodic sets (see Section 4.2).

For the lattice problems, our algorithmic solution uses the possibility to obtain a complete system of $\mathsf{GL}_d(\mathbb{Z})$-inequivalent Delone triangulations with vertex-set $\mathbb{Z}^d$. By Theorem 4.13 we can also obtain a complete system of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone triangulations with rational standard periodic vertex-set

$$\Lambda_{\boldsymbol{t}} = \bigcup_{i=1}^{m} \boldsymbol{t}_i + \mathbb{Z}^d,$$

where $\boldsymbol{t} = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_{m-1}) \in \mathbb{Q}^{d\times(m-1)}$ and (without loss of generality) $\boldsymbol{t}_m = \mathbf{0}$.

For each Delone subdivision with vertex-set $\Lambda_{\boldsymbol{t}}$, there exists a finite system of $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent Delone polytopes. Recall from Section 4.2.3 that two Delone polytopes $P$ and $P'$ (with vertices in $\Lambda_{\boldsymbol{t}}$) are called $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-equivalent (with respect to a given $Q \in \mathcal{S}^d_{>0}$ or a finite subgroup $G \subset \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$), if there exists a $U \in \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ ($U \in \operatorname{Aut} Q$ or $U \in G$) and a $\boldsymbol{v} \in \Lambda_{\boldsymbol{t}} - \Lambda_{\boldsymbol{t}}$ with $P = \boldsymbol{v} + UP'$. With this terminology, Proposition 5.4 has the following generalization. Recall that for $m = 1$ we set $\Lambda_{\boldsymbol{t}} = \mathbb{Z}^d$ and $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z}) = \mathsf{GL}_d(\mathbb{Z})$.

PROPOSITION 5.5. *Let $d, m \geq 1$ and $\boldsymbol{t} \in \mathbb{R}^{d\times(m-1)}$. Let $Q \in \mathcal{S}^d_{>0}$ be a PQF and let $P_1, \ldots, P_n$ be a representative system of $d$-dimensional Delone polytopes in $\operatorname{Del}_{\Lambda_{\boldsymbol{t}}}(Q)$, which are $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-inequivalent with respect to $Q$. For every $P_i$ choose a $d$-dimensional simplex $L_i$ with* $\operatorname{vert} L_i \subseteq \operatorname{vert} P_i$. *Then for $\beta > 0$*

$$\mu_{\Lambda_{\boldsymbol{t}}}(Q) \leq \beta \quad \Longleftrightarrow \quad \begin{pmatrix} \boxed{\mathrm{BR}_{L_1,\beta}(Q)} & 0 & 0 & \ldots & 0 \\ 0 & \boxed{\mathrm{BR}_{L_2,\beta}(Q)} & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \boxed{\mathrm{BR}_{L_n,\beta}(Q)} \end{pmatrix} \succeq 0.$$

As in the lattice case, using this proposition, for each Delone subdivision $\mathcal{D}$ with vertices in $\Lambda_{\boldsymbol{t}}$, we can solve the determinant maximization problem (5.8) to obtain a local optimum for the covering problem. Respectively, we can solve the semidefinite programming problem (5.9), to obtain a local optimum for the lattice-covering problem In this way, for a fixed rational $\boldsymbol{t}$, we can obtain the best covering and packing-covering sets among all periodic sets $A\Lambda_{\boldsymbol{t}}$ with $A \in \mathsf{GL}_d(\mathbb{R})$. By this we have a theory which enables us to systematically search for good or even best periodic (possibly non-lattice) coverings or packing-coverings.

5.1.4.4. *Extension to linear subspaces $T \subseteq \mathcal{S}^d$.* The extension of Voronoi's theory in Section 4.3 dealt with the restriction to PQFs lying in a linear subspace $T \subseteq \mathcal{S}^d$. This can be applied to the classical lattice theory as well as to the extension to standard periodic sets $\Lambda_{\boldsymbol{t}}$ mentioned above.

Assume $(A_1, \dots, A_{\dim T})$ is a basis of $T$. Then every $Q \in T$ can be written as $Q = \sum_{i=1}^{\dim T} x_i A_i$ with unique parameters $x_1, \dots, x_{\dim T}$. The inner product $(\cdot, \cdot)$ defined by $(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{y}^t Q \boldsymbol{z}$ (used in Propositions 5.1 and 5.4) can be expressed as

$$(\boldsymbol{y}, \boldsymbol{z}) = \langle Q, \boldsymbol{y}\boldsymbol{z}^t \rangle = \sum_{i=1}^{\dim T} x_i \langle A_i, \boldsymbol{y}\boldsymbol{z}^t \rangle,$$

hence as a linear combination of the parameters $x_i$. Thus instead of $\binom{d+1}{2}$ parameters we now deal with $\dim T$ parameters. In (5.5), the matrix $G(\boldsymbol{x})$ is $\sum_{i=1}^{\dim T} x_i A_i$. Depending on $T$ this may reduce the size of the optimization problems tremendously.

We can reduce the size of the optimization problems further, in case $T = T_G$ is an invariant subspace for some finite group $G \subset \mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$. Because in this case, every $Q \in T$ as well as their Delone subdivisions are invariant with respect to the elements in $G$, we can apply Proposition 5.4 and reduce the number of blocks in the used block diagonal matrices (by considering only one block per $G$-equivalence class of Delone polytopes).

**5.1.5. A covering and a packing-covering optimizer.** Based on the program `rmd` (*rigorous MAXDET*) we developed two software tools: `coop`, a *covering optimizer*, and `pacoop`, a *packing-covering optimizer*. The programs, together with a short tutorial can be obtained from the web page [**227**]. They allow to approximate — mathematically rigorously — the optimum covering density and the packing-covering constant with respect to a given $T$-secondary cone.

To give a short example let us consider possible input and output for `coop` (see Figure 1). The first line of the input file contains the dimension, here 2. The second block contains the number of simplices, here 1, which follow to describe the Delone triangulation (or more general, simplices that describe the Delone subdivision). All simplices are assumed to contain the origin. So here, the block describes the simplex with vertices $(0,0)$, $(1,0)$ and $(1,1)$. The second block describes a basis for the $T$-space under consideration. Here we have a 2-dimensional space spanned by the two symmetric matrices, which are given in lower triangular form. The third block contains linear inequalities for the coefficients of forms with respect to the basis given by the $T$-space. They describes the $T$-secondary cone under consideration. In this case we have 2 inequalities for 2 variables $x_1$ and $x_2$. They have to be read as $-x_2 \geq 0$ and $x_1 + x_2 \geq 0$. The last two entries give some technicalities for the involved interior point methods. The first number denotes the maximum number of iterations and the second number gives a size of the duality gap one wants to obtain. This gap gives information on the quality of the approximation of the optimal covering density with respect to the given secondary cone. The output of a call of `coop` with this input data is shown on the right in Figure 1. The entry after `minimizer_approx` gives a rational approximation (in homogeneous coordinates) of

| **INPUT:** | **OUTPUT:** |
|---|---|

```
2                          ***********
                           * SUMMARY *
1                          ***********
1 0
1 1                        * minimizer_approx = [10000000000000000000
                                                  29999999999696198571
2                                                -15000000000064459549]
1
0 1                        * theta_lower_bound ~ 1.209199576
0
1 0                        * theta_upper_bound ~ 1.209199576

2                          * duality_gap ~ 1.493172253e-11
0 -1
1 1                        * Thanks for your patience...

500

1e-10
```

FIGURE 1. Example input (left) and partial output (right) of `coop`.

a form attaining optimality within the given secondary cone. Here the answer can be interpreted to approximately

$$2.9999999999696198571 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1.5000000000064459549 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is very close to a PQF representing a basis of the hexagonal lattice. In cases where the solution is "obvious" as here one may guess the correct PQF and check ("prove"), using the software `rmd`, whether or not it actually is the solution for the given input data.

## 5.2. Local analysis of lattice coverings

Since we are dealing with convex optimization problems, we can extract some structural information about uniqueness and invariance properties of locally optimal solutions. On the one hand these help us to identify exact coordinates of optimal solutions. On the other hand they allow us to decide whether we have found an isolated locally optimal solution or not. Whereas in this section we concentrate on a local analysis of lattice covering optima (density minima), we complement the discussion in Section 5.6 by analyzing possible lattice covering maxima.

**5.2.1. Properties of local optima.** Most of the desired properties follow from the special structure of sets with PQFs $Q$ attaining certain fixed values $\mu(Q)$, $\lambda(Q)$ or $\det Q$. Let $L = \operatorname{conv}\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\} \subseteq \mathbb{R}^d$ denote a $d$-dimensional simplex and $\mathcal{D}$ a Delone triangulation. Then for $\beta > 0$ we consider the sets

$$\begin{aligned}\mathcal{B}_{L,\leq\beta} &= \{Q \in \mathcal{S}^d_{>0} : |\mathrm{BR}_{L,\beta}(Q)| \geq 0\},\\ \mathcal{B}_{L,=\beta} &= \{Q \in \mathcal{S}^d_{>0} : |\mathrm{BR}_{L,\beta}(Q)| = 0\},\\ \mathcal{W}_{\mathcal{D},\leq\beta} &= \bigcap_{L\in\mathcal{D}} \mathcal{B}_{L,\leq\beta} \cap \mathbf{\Delta}(\mathcal{D}).\end{aligned}$$

Hence, by Corollary 5.2 and Proposition 5.4 the PQFs $Q \in \mathcal{W}_{\mathcal{D},\leq\beta}$ are those with $\mu(Q) \leq \beta$ in $\mathbf{\Delta}(\mathcal{D})$. Further, for $\alpha > 0$ and $\lambda > 0$ we consider

$$\mathbf{D}_\alpha = \{Q \in \mathcal{S}^d_{>0} : \det Q \geq \alpha\}$$

and the Ryshkov polyhedron

$$\mathcal{P}_\lambda = \{Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq \lambda\}.$$

The following is known (see [**77**], [**211**]) and was partially discussed in previous sections:

Proposition 5.6. *Let $L = \operatorname{conv}\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\} \subseteq \mathbb{R}^d$ be a d-dimensional simplex, $\mathcal{D}$ a Delone triangulation with vertex-set $\mathbb{Z}^d$, $D > 0$ and $\lambda > 0$. Then*

1. *$\mathcal{B}_{L,\leq\beta}$, and therefore $\mathcal{W}_{\mathcal{D},\leq\beta}$, is convex and bounded.*
2. *$\mathcal{B}_{L,=\beta}$ is a smooth (regular) hypersurface.*
3. *$\mathbf{D}_\alpha$ is unbounded, strictly convex and has a smooth boundary.*
4. *$\mathcal{P}_\lambda$ is unbounded and convex and has a piecewise linear boundary.*
5. *$\mathbf{D}_\alpha$ and $\mathcal{P}_\lambda$ are invariant with respect to the action of $\mathsf{GL}_d(\mathbb{Z})$ on $\mathcal{S}^d_{>0}$.*

As a consequence of these properties, it is not hard to derive the following proposition (see [**77**], [**211**]).

Proposition 5.7. *Let $\mathcal{D}$ be a Delone triangulation with vertex set $\mathbb{Z}^d$.*

1. *The set of PQFs in $\overline{\mathbf{\Delta}(\mathcal{D})}$ attaining $\min\limits_{Q\in\overline{\mathbf{\Delta}(\mathcal{D})}} \Theta(Q)$ is a single PQF, together with all of its positive multiples. It is invariant with respect to the action of $\mathcal{D}$'s* automorphism group
$$\operatorname{Aut}(\mathcal{D}) = \{U \in \mathsf{GL}_d(\mathbb{Z}) : U\mathcal{D} = \mathcal{D}\}.$$
2. *The set of PQFs in $\overline{\mathbf{\Delta}(\mathcal{D})}$ attaining $\min\limits_{Q\in\overline{\mathbf{\Delta}(\mathcal{D})}} \gamma(Q)$ is convex and contains a subset which is invariant with respect to* $\operatorname{Aut}(\mathcal{D})$.

A first, but non-geometric proof of property (1) was given by Barnes and Dickson [**27**]. There they also observed, as Ryshkov [**211**] did for the lattice packing-covering problem, that it suffices to optimize among all PQFs whose automorphism group contains the group $\operatorname{Aut}(\mathcal{D})$. This was one of the motivations to develop the theory of $G$-secondary cones, described in Chapter 4.

Corollary 5.8. *If $Q$ is a local optimum of the lattice covering or lattice packing-covering problem among all PQFs in $\mathbf{\Delta}(\mathcal{D})$ whose automorphism group contains* $\operatorname{Aut}(\mathcal{D})$*, then $Q$ is a locally optimal solution.*

Note that the statement of Corollary 5.8 holds only if $Q$ lies in the *relative interior* of the closed cone $\overline{\mathbf{\Delta}(\mathcal{D})}$. For local optima on the boundary of secondary cones we may apply the following trivial proposition.

PROPOSITION 5.9. *A PQF $Q$ is a locally optimal solution with respect to $\Theta$ or $\gamma$ if and only if it is an optimal solution for all Delone triangulations $\mathcal{D}$ with $Q \in \overline{\mathbf{\Delta}(\mathcal{D})}$.*

Computational experiments suggest that there exist two locally optimal solutions with respect to $\gamma$ in dimension 5 which both lie on the boundary of some secondary cones (see the proof of Theorem 5.25). In dimension 5 and lower every locally optimal solution with respect to $\Theta$ lies in the interior of some secondary cone. However, in higher dimensions there exist locally optimal solutions lying on the boundary. The Leech lattice gives such an example as we show in Section 5.5.

**5.2.2. Rigorous certificates.** Using convex optimization software to solve the covering or packing-covering problem, we are often limited to determining a certain certified range (see Section 5.1.2) in which the optimum value is attained. To use Proposition 5.9 it is desirable to decide computationally whether or not the optimum is attained on the boundary or even on a specific facet of $\overline{\mathbf{\Delta}(\mathcal{D})}$. In many cases this is possible by using the following proposition which is a simple consequence of the convexity of determinant maximization problems.

PROPOSITION 5.10. *Consider the determinant maximization problem*

$$\begin{array}{ll} \textbf{\textit{minimize}} & f(\boldsymbol{x}) \\ \textbf{\textit{subject to}} & G(\boldsymbol{x}) \succ 0 \text{ and } F(\boldsymbol{x}) \succeq 0, \end{array} \tag{5.10}$$

*where $f(\boldsymbol{x}) = \boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})$.*

(1) *Let $\boldsymbol{y}_1$ be the optimal solution of (5.10), and let $\boldsymbol{y}_2$ be the optimal solution of (5.10) with the additional constraint $\tilde{F}(\boldsymbol{x}) \succeq 0$. If $f(\boldsymbol{y}_1) < f(\boldsymbol{y}_2)$, then $\tilde{F}(\boldsymbol{y}_2) = 0$.*

(2) *Let $\boldsymbol{y}_1$ be an optimal solution of (5.10) with the additional linear constraint $\boldsymbol{n}^t\boldsymbol{x} \geq 0$, and let $\boldsymbol{y}_2$ be an optimal solution of (5.10) with the additional linear constraint $\boldsymbol{n}^t\boldsymbol{x} \leq 0$.*
   (a) *If $f(\boldsymbol{y}_1) < f(\boldsymbol{y}_2)$, then $\boldsymbol{n}^t\boldsymbol{y}_2 = 0$.*
   (b) *If $f(\boldsymbol{y}_1) > f(\boldsymbol{y}_2)$, then $\boldsymbol{n}^t\boldsymbol{y}_2 < 0$.*

Let $Q_{\text{opt}}$ denote a PQF with $\mu(Q_{\text{opt}}) = \beta$ attaining an optimum in $\overline{\mathbf{\Delta}(\mathcal{D})}$ for $\Theta$. The lattice packing-covering case is similar. Let $l$ and $u$ be a lower and an upper bound for a minimum of $f(Q) = -\log\det Q$. Here and in the sequel we use $\mu_{\mathcal{D}}(Q) \leq \beta$ as an abbreviation for the linear matrix inequality in Proposition 5.4 guaranteeing that the circumradius (with respect to $Q$) of all the simplices of the triangulation $\mathcal{D}$ is at most $\beta^{\frac{1}{2}}$ ($\beta > 0$).

The first part of Proposition 5.10 gives a sufficient criterion for $Q_{\text{opt}}$ lying on the boundary of $\overline{\mathbf{\Delta}(\mathcal{D})}$. Suppose the upper bound of $f$ on $\{Q \in \mathcal{S}^d_{>0} : \mu_{\mathcal{D}}(Q) \leq \beta\}$ is smaller than $l$. Then we have a certificate for $Q_{\text{opt}} \in \operatorname{bd} \overline{\mathbf{\Delta}(\mathcal{D})}$.

The first half of the second part of Proposition 5.10 gives a sufficient criterion for $Q_{\text{opt}}$ lying on a specific facet $F$ of $\overline{\mathbf{\Delta}(\mathcal{D})}$. Let $H$ denote a hyperplane containing $F$ and let $H^-$, $H^+$ be the closed halfspaces containing $\overline{\mathbf{\Delta}(\mathcal{D})}$, respectively not containing $\overline{\mathbf{\Delta}(\mathcal{D})}$. If the upper bound of $f$ on

$$\{Q \in \mathcal{S}^d_{>0} : Q \in H^+ \text{ and } \mu_{\mathcal{D}}(Q) \leq \beta\}$$

is smaller than $l$, then we have a certificate for $Q_{\text{opt}} \in F$.

The second half of the second part of Proposition 5.10 gives a sufficient criterion for $Q_{\text{opt}}$ not lying on a specific facet $F$ of $\overline{\mathbf{\Delta}(\mathcal{D})}$. If the lower bound of $f$ on

$$\{Q \in \mathcal{S}^d_{>0} : Q \in H^+ \text{ and } \mu_{\mathcal{D}}(Q) \le \beta\}$$

is larger than $u$, then we have a certificate for $Q_{\text{opt}} \notin F$. Clearly, such a certificate for all facets of $\overline{\mathbf{\Delta}(\mathcal{D})}$ gives a certificate for $Q_{\text{opt}} \in \mathbf{\Delta}(\mathcal{D})$.

Note that these certificates give mathematical rigorous proofs when we proceed as described at the end of Section 5.1.2.

**5.2.3. Outer normal cones.** Next we assume a PQF $Q$ is given and we want to decide computationally if it is a locally optimal solutions to the lattice covering or lattice packing-covering problem. Before considering locally optimal solution we have to do some local analysis. Here, we shall compute the normal cones of the sets $\mathcal{W}_{\mathcal{D},\le\beta}$, $\mathcal{P}_\lambda$ and $\mathbf{D}_\alpha$ at a given PQF $Q$. As a general reference to the basic concepts in convex and differential geometry used in the sequel we refer to the book [**222**] by Schneider.

We consider $\mathcal{S}^d$ as Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let

$$H = \{S \in \mathcal{S}^d : \langle N, S \rangle = \alpha\}$$

be a hyperplane with *normal vector* $N \in \mathcal{S}^d \backslash \{\mathbf{0}\}$ and $C$ a convex set with boundary point $Q$. Then $H$ is said to be a supporting hyperplane of $C$ at $Q$ with outer normal vector $N$, if

$$C \subseteq H^- = \{S \in \mathcal{S}^d : \langle N, S \rangle \le \langle N, Q \rangle\}.$$

The *(outer) normal cone* of $C$ at $Q$ is then given by all outer normal vectors of supporting hyperplanes at $Q$ together with the zero vector. Clearly the normal cones are convex. A hyperplane $H$ is called a separating hyperplane for two convex sets with a common boundary point $Q$, if it is a supporting hyperplane of both sets, but with opposite outer normal vectors. Such a hyperplane exists if and only if the corresponding normal cones $\mathcal{N}_1$ and $\mathcal{N}_2$ at $Q$ satisfy $-\mathcal{N}_1 \cap \mathcal{N}_2 \ne \emptyset$.

Let us now compute the normal cones of the sets $\mathcal{W}_{\mathcal{D},\le\beta}$, $\mathcal{P}_\lambda$ and $\mathbf{D}_\alpha$ at some PQF $Q$.

PROPOSITION 5.11. *Let $\mathcal{D}$ be a Delone triangulation and $Q \in \mathcal{W}_{\mathcal{D},\le\beta}$ a PQF with $\mu(Q) = \beta > 0$. Then the normal cone of $\mathcal{W}_{\mathcal{D},\le\beta}$ at $Q$ is equal to*

$$-\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\},$$

*where*

$$g_{L,\beta}(Q) = \operatorname{grad} |\operatorname{BR}_{L,\beta}|(Q)$$

*denotes the gradient of the regular surface $\mathcal{B}_{L,=\beta}$ at $Q$.*

PROOF. First, $-g_{L,\beta}(Q) \ne \mathbf{0}$ exists and is the unique outer normal vector of $\mathcal{B}_{L,\le\beta}$ at $Q$, since $\mathcal{B}_{L,=\beta}$ is a regular surface defined by the polynomial equation $|\operatorname{BR}_{L,\beta}(Q)| = 0$. Second, we see that $\mathcal{W}_{\mathcal{D},\le\beta}$, in a sufficiently small neighborhood of $Q$, is equal to the intersection $\bigcap_{Q\in\mathcal{B}_{L,=\beta}} \mathcal{B}_{L,\le\beta}$. Therefore, the normal cone of $\mathcal{W}_{\mathcal{D},\le\beta}$ at $Q$ is equal to $\operatorname{cone}\{-g_{L,\beta}(Q) : Q \in \mathcal{B}_{L,=\beta}\}$ (see [**222**, Th. 2.2.1]) and the assertion follows. $\square$

From Section 3.1.3 and the proof of Theorem 2.3 we obtain the outer normal cones for $\mathbf{D}_\alpha$ and the Ryshkov polyhedra $\mathcal{P}_\lambda$. For latter we obtain exactly the negative of the Voronoi domain $\mathcal{V}(Q) = \operatorname{cone}\{\boldsymbol{v}\boldsymbol{v}^t : \boldsymbol{v} \in \operatorname{Min} Q\}$.

PROPOSITION 5.12. *The outer normal cone of* $\mathbf{D}_{\det Q}$ *at* $Q$ *is given by*

$$-\operatorname{cone}\{Q^{-1}\}.$$

PROPOSITION 5.13. *The outer normal cone of* $\mathcal{P}_{\lambda(Q)}$ *at* $Q$ *is given by* $-\mathcal{V}(Q)$.

**5.2.4. Necessary and sufficient conditions for local optima.** Based on the information about the outer normal cones derived in the previous section, we can give sufficient conditions for local optimality. For this we distinguish the cases where $Q$ lies in the interior or boundary of the closure of a secondary cone of some Delone triangulation.

5.2.4.1. *Interior Cases.* Because of the convexity of $\mathbf{D}_{\det Q}$, $\mathcal{P}_{\lambda(Q)}$ and $\mathcal{W}_{\mathcal{D},\leq\beta}$ a separating hyperplane at a PQF $Q$ of $\mathcal{W}_{\mathcal{D},\leq\beta}$ with $\mu(Q)=\beta$ and one of the other two sets yields a necessary and sufficient condition for $Q$ to be a locally optimal solution to either the lattice covering or the lattice packing-covering problem. Therefore a necessary and sufficient condition can be derived from the normal cones at $Q$ with respect to these sets.

PROPOSITION 5.14. *Let* $\mathcal{D}$ *be a Delone triangulation with vertex-set* $\mathbb{Z}^d$, *and let* $Q \in \mathbf{\Delta}(\mathcal{D})$ *be a PQF with* $\mu(Q)=\beta>0$.

(1) *(*[**27**]*) Then* $Q$ *is a locally optimal solution to the lattice covering problem if and only if*

$$\operatorname{cone}\{Q^{-1}\} \cap -\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\} \neq \emptyset.$$

(2) *Then* $Q$ *is a locally optimal solution to the lattice packing-covering problem if and only if*

$$\mathcal{V}(Q) \cap -\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\} \neq \emptyset.$$

Recall that $Q$ is called eutactic, if $Q^{-1} \in \operatorname{relint}\operatorname{cone}\{\boldsymbol{v}\boldsymbol{v}^t : \boldsymbol{v} \in \operatorname{Min} Q\}$. Thus combining the two statements of the proposition, we obtain a criterion for a unique local optimum with respect to the lattice packing-covering problem.

COROLLARY 5.15. *Let* $\mathcal{D}$ *be a Delone triangulation with vertex-set* $\mathbb{Z}^d$. *Then* $Q \in \mathbf{\Delta}(\mathcal{D})$ *is a unique locally optimal solution to the lattice packing-covering problem, if* $Q$ *is eutactic and a locally optimal solution to the lattice covering problem.*

EXAMPLE 5.16. We can use Corollary 5.15 to show that $\mathsf{A}_d^*$, $d \geq 2$, with $\gamma(\mathsf{A}_d^*) = \sqrt{\frac{d+2}{3}}$ is locally optimal for the lattice packing-covering problem. Ryshkov [**211**] gave another proof of this fact. The lattice $\mathsf{A}_d^*$ is known to give a locally optimal lattice covering (see [**112**], [**113**], [**42**]). A quadratic form $Q_{\mathsf{A}_d^*}$ associated with $\mathsf{A}_d^*$ is

$$Q_{\mathsf{A}_d^*} = \begin{pmatrix} d & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & d \end{pmatrix},$$

with $\mu(Q_{\mathsf{A}_d^*}) = \frac{d(d+2)}{12}$ and $\lambda(Q_{\mathsf{A}_d^*}) = d$. The set $\operatorname{Min} Q_{\mathsf{A}_d^*}$ contains exactly $2(d+1)$ elements, namely the standard basis vectors $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d$, their negatives and $\pm\sum_{i=1}^d \boldsymbol{e}_i$

(see [**69**]). Thus in particular

$$(d+1)\cdot Q_{\mathsf{A}_d^*}^{-1} = Q_{\mathsf{A}_d} = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix} = \sum_{\boldsymbol{v}\in \operatorname{Min} Q_{\mathsf{A}_d^*}} \boldsymbol{v}\boldsymbol{v}^t.$$

Hence, $Q_{\mathsf{A}_d^*}$ is eutactic because $Q_{\mathsf{A}_d^*}^{-1} \in \operatorname{relint} \mathcal{V}(Q_{\mathsf{A}_d^*})$ and therefore the assertion follows.

5.2.4.2. *Boundary cases.* In Proposition 5.14 we assume that $\mathcal{D} = \operatorname{Del}_{\mathbb{Z}^d}(Q)$ is a Delone triangulation. If this is not the case, the situation becomes more complicated, in particular for the lattice packing-covering problem. Assume $Q$ is on the boundary of a secondary cone of a Delone triangulation. Then we need to replace

$$-\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\}$$

by a more general expression.

We not only have to consider one secondary cone, but closures of all in which $Q$ lies. For this let $\mathcal{D} = \operatorname{Del}_{\mathbb{Z}^d}(Q)$ and

$$\mathcal{W}_{\mathcal{D},\beta} = \bigcup_{\mathcal{D}'<\mathcal{D}} \{Q \in \overline{\boldsymbol{\Delta}(\mathcal{D}')} : Q \in \mathcal{B}_{L,\leq\beta} \text{ for all } L \in \mathcal{D}'\},$$

where $\mathcal{D}' < \mathcal{D}$ denotes that $\mathcal{D}'$ is a Delone triangulations refining $\mathcal{D}$. This set is a subset of $\{Q \in \mathcal{S}^d_{>0} : \mu(Q) \leq \beta\}$. We require that $\mathcal{W}_{\mathcal{D},\beta}$ is *separable at* $Q$, that is, there exists a supporting hyperplane of $\mathcal{W}_{\mathcal{D},\beta}$ through $Q$. This is in particular the case, if there exists a small $r > 0$ such that $(Q + rB) \cap \mathcal{W}_{\mathcal{D},\beta}$ is convex, where $B = \{X \in \mathcal{S}^d : \langle X, X\rangle = 1\}$ is the unit ball in $\mathcal{S}^d$. In this case, the outer normal cone of $\mathcal{W}_{\mathcal{D},\beta}$ at $Q$ is

$$-\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D}' < \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\}.$$

As a consequence we obtain the following criterion for local optimality.

PROPOSITION 5.17. *Let $\mathcal{D}$ be a Delone subdivision with vertex-set $\mathbb{Z}^d$ and $Q \in \boldsymbol{\Delta}(\mathcal{D})$ with $\mu(Q) = \beta$. Then*

(1) *$Q$ is a locally optimal solution to the lattice covering problem, if and only if $\mathcal{W}_{\mathcal{D},\beta}$ is separable at $Q$ and*

$$\operatorname{cone}\{Q^{-1}\} \cap -\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D}' < \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\} \neq \emptyset$$

(2) *$Q$ is a locally optimal solution to the lattice packing-covering problem, if $\mathcal{W}_{\mathcal{D},\beta}$ is separable at $Q$ and*

$$\mathcal{V}(Q) \cap -\operatorname{cone}\{g_{L,\beta}(Q) : L \in \mathcal{D}' < \mathcal{D} \text{ and } Q \in \mathcal{B}_{L,=\beta}\} \neq \emptyset.$$

In case of the lattice packing-covering problem the "only if" part is missing, because we cannot exclude the case of a locally optimal solution $Q'$with $\mathcal{W}_{\mathcal{D},\beta}$ not being separable at $Q'$. This is due to the fact that the boundary of the Ryshkov polyhedron $\{Q \in \mathcal{S}^d_{>0} : \lambda(Q) \geq \lambda(Q')\}$ is not smooth in contrast to the boundary of $\{Q \in \mathcal{S}^d_{>0} : \det Q \geq \det Q'\}$.

This phenomenon seems to happen to PQFs associated to the root lattice $\mathsf{E}_8$. This lattice is known to give a globally optimal solution to the lattice packing problem, but not a locally optimal solution to the lattice covering problem (see Section 5.5.5). Nevertheless, computational experiments support the **conjecture** that the root lattice $\mathsf{E}_8$ gives a locally optimal solution for the lattice packing-covering problem. Note that Zong [**272**] even conjectured that $\mathsf{E}_8$ gives the unique globally optimal solution to the lattice packing-covering problem in dimension 8.

**5.2.5. Conditions for isolated local optima.** Locally optimal solutions to the lattice packing-covering problem may not be isolated optima, in contrast to solutions to the lattice covering problem. To determine computationally if a given locally optimal solution $Q$ is isolated, we have to check if there exists a segment $[Q, Q']$ which lies in the boundary of $\mathcal{P}_{\lambda(Q)}$ and in the boundary of $\mathcal{W}_{\mathcal{D},\leq\beta}$ at $Q$. For simplicity we only deal with the case where $\mathrm{Del}_{\mathbb{Z}^d}(Q)$ is a triangulation.

PROPOSITION 5.18. *Let $\mathcal{D}$ be a Delone triangulation with vertex-set $\mathbb{Z}^d$ and let $Q \in \mathbf{\Delta}(\mathcal{D})$ be a locally optimal solution to the lattice packing-covering problem with $\mu(Q) = \beta > 0$. Then $Q$ is not an isolated local optimum if and only if there exists an $S \in \mathcal{S}^d$ with*

(1) $\langle \boldsymbol{v}\boldsymbol{v}^t, S\rangle = S[\boldsymbol{v}] \geq 0$ *for all* $\boldsymbol{v} \in \operatorname{Min} Q$,
(2) $\langle g_{L,\beta}(Q), S\rangle \geq 0$ *for all* $L \in \mathcal{D}$ *with* $Q \in \mathcal{B}_{L,=\beta}$,
(3) $h_L(Q)S = \mathbf{0}$ *for all* $L \in \mathcal{D}$ *with* $Q \in \mathcal{B}_{L,=\beta}$ *and* $\langle g_{L,\beta}(Q), S\rangle = 0$.

PROOF. The first condition says that there exists an $\epsilon_1 > 0$ so that the segment $[Q, Q + \epsilon_1 S]$ lies in $\mathcal{P}_{\lambda(Q)}$ because $\mathcal{P}_{\lambda(Q)}$ has a piecewise linear boundary. The first condition together with the second condition plus the local optimality of $Q$ are equivalent to the fact that $S$ is in the tangent space of $\mathcal{B}_{L,\leq\beta}$ at $Q$ and in the tangent space of $\mathcal{P}_{\lambda(Q)}$ at $Q$. Thus $Q+\epsilon_2 S$ is in the boundary of $\mathcal{W}_{\mathcal{D},\leq\beta}$ for sufficiently small $\epsilon_2 > 0$ if and only if the corresponding smooth hypersurfaces $\mathcal{B}_{L,=\beta}$ have curvature 0 in direction $S$. This is equivalent to $S$ being an eigenvector of eigenvalue 0 of the Hessian $h_L(Q) = \operatorname{hess}|\operatorname{BR}_{L,\beta}|(Q)$ which is the third condition. Hence, all three conditions are fulfilled if and only if there exists a segment $[Q, Q']$ in the boundary of the two sets $\mathcal{P}_{\lambda(Q)}$ and $\mathcal{W}_{\mathcal{D},\leq\beta}$, where $Q' = Q + \min(\epsilon_1, \epsilon_2)S$. $\square$

To this point we do not know whether or not there exist a locally optimal solution to the lattice packing-covering problem which are not isolated. Their existence would for example follow if there would exist a locally optimal solution to the lattice packing-covering problem which does not have all the symmetries of the corresponding Delone subdivision.

**5.2.6. Local lower bounds via moments of inertia.** In this section we give simple and efficiently computable local lower bounds for the lattice covering density and the lattice packing-covering constant. These bounds are "local" in the sense that they only apply to those PQFs lying in the topological closure of the secondary cone of a given part of a Delone triangulation (see Section 4.4 for the concept of secondary cones of polyhedral complexes). In order to compute the bounds we only need to know the coordinates of the considered simplices. These bounds are therefore useful tools in a heuristic search for "good" PQFs. Sometimes they can be used to obtain a proof for local optimality of a lattice covering, as we shall see in Section 5.5 for the Leech lattice. For complete Delone triangulations, the method goes back to Ryshkov and Delone. It is called the *method of the moments of inertia* because the central idea in its proof is analogous to the Parallel Axis Theorem of Steiner in classical mechanics (see [**3**]).

Let $L = \operatorname{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}\} \subseteq \mathbb{R}^d$ be a simplex in the $d$-dimensional Euclidean space with inner product given by the PQF $Q$. Its *centroid* is $\boldsymbol{m} = \frac{1}{d+1}\sum_i \boldsymbol{v}_i$. Let

$\boldsymbol{c}$ be the center of its circumsphere and let $r$ be its circumradius. Using Apollonius' formula (see [**38**, §9.7.6]) we get

$$R^2 = Q[\boldsymbol{c} - \boldsymbol{m}] + \frac{1}{(d+1)^2} \sum_{k \neq l} Q[\boldsymbol{v}_k - \boldsymbol{v}_l].$$

Based on this relation we can bound the inhomogeneous minimum.

PROPOSITION 5.19. *We consider a collection of Delone simplices*

$$L_1 = \operatorname{conv}\{\boldsymbol{v}_{1,1}, \ldots, \boldsymbol{v}_{1,d+1}\}, \ldots, L_n = \operatorname{conv}\{\boldsymbol{v}_{n,1}, \ldots, \boldsymbol{v}_{n,d+1}\}$$

*of $Q$ with vertices in $\mathbb{Z}^d$ and with circumradii $R_1, \ldots, R_n$. Then the inhomogeneous minimum is bounded by*

$$\mu(Q) \geq \max_i R_i^2 \geq \frac{1}{n(d+1)^2} \sum_i \sum_{k \neq l} Q[\boldsymbol{v}_{i,k} - \boldsymbol{v}_{i,l}]. \tag{5.11}$$

The proof is straightforward. We can use this proposition to get local lower bounds for the covering density and the packing-covering constant of PQFs having $L_1, \ldots, L_n$ as Delone simplices. We fix the determinant of $Q$ and minimize the right hand side of (5.11), which is a linear function $\langle F, \cdot \rangle$ with the PQF

$$F = \frac{1}{n(d+1)^2} \sum_i \sum_{k \neq l} (\boldsymbol{v}_{i,k} - \boldsymbol{v}_{i,l})(\boldsymbol{v}_{i,k} - \boldsymbol{v}_{i,l})^t. \tag{5.12}$$

Given an arbitrary PQF $F$ defining a linear function $\langle F, \cdot \rangle$, we can determine its minimum on the determinant-$\alpha$-surface:

PROPOSITION 5.20. *Let $\alpha > 0$. A linear function $f(Q') = \langle F, Q' \rangle$ with a PQF $F$ has a unique minimum on the determinant-$\alpha$-surface*

$$\{Q \in \mathcal{S}^d_{>0} : \det Q = \alpha\}.$$

*Its value is $d\sqrt[d]{\alpha \det F}$ and the minimum is attained by the PQF $\sqrt[d]{\alpha \det F} F^{-1}$.*

PROOF. A proof follows directly from the fact that $F$ is the normal of the smooth determinant-$\alpha$-surface at $\sqrt[d]{\alpha \det F} F^{-1}$ (see (3.15)). $\square$

Together, Proposition 5.19 and Proposition 5.20 yield a bound for the lattice covering density.

PROPOSITION 5.21. *As in Proposition 5.19, let $L_1, \ldots, L_n$ be a collection of Delone simplices of a PQF $Q$. Then*

$$\Theta(Q) \geq \sqrt{d^d \det F} \cdot \operatorname{vol}(B^d),$$

*with the PQF $F$ in* (5.12).

PROOF. By definition $\Theta(Q) = \sqrt{\frac{(\mu(Q))^d}{\det Q}} \operatorname{vol}(B^d)$. We use the bound (5.11) for $\mu(Q)$ and choose $\alpha = \det Q$ in Proposition 5.20. $\square$

For a local lower bound on the lattice packing-covering constant we minimize the linear function $\langle F, \cdot \rangle$ over all PQFs with a fixed homogeneous minimum. In the above, we just replace the determinant-$\alpha$-surface (the boundary of $\mathbf{D}_\alpha$) by the *homogeneous-minimum-$\lambda$-surface*

$$\{Q \in \mathcal{S}^d_{>0} : \lambda(Q) = \lambda\},$$

the boundary of the Ryshkov polyhedron $\mathcal{P}_\lambda$. Using the fact that the Voronoi cone $\mathcal{V}(Q)$ is the normal cone of $\mathcal{P}_\lambda$ at $Q$, we find:

PROPOSITION 5.22. *Let $\lambda > 0$. A linear function $f(Q') = \langle F, Q' \rangle$ with a PQF $F$ has a unique minimum on the* homogeneous-minimum-$\lambda$-surface. *It is attained by $Q_F$ if and only if $F \in \mathcal{V}(Q_F)$.*

Then a similar argument as in the proof of Proposition 5.21 yields a local lower bound for the packing-covering constant:

PROPOSITION 5.23. *As in Proposition 5.19, let $L_1, \dots, L_n$ be a collection of Delone simplices of a PQF $Q$. Then*

$$\gamma(Q) \geq 2\sqrt{\frac{\langle F, Q_F \rangle}{\lambda(Q_F)}}$$

*with the PQF $F$ in* (5.12) *and with $Q_F$ such that $F \in \mathcal{V}(Q_F)$.*

Note, if $Q$ is eutactic and $F = Q^{-1}$, then Proposition 5.23 is immediately applicable with $Q_F = Q$. In Section 5.5.4 we use this fact and apply Proposition 5.23 to the Leech lattice.

## 5.3. New, best known coverings and packing-coverings

In Section 5.1.4 we developed algorithms for the solution of the lattice covering and the lattice packing-covering problem. Here, we want to demonstrate that these algorithms are not purely of theoretical interest. We implemented the algorithms in C++, using the package `MAXDET` [**280**] of Wu, Vandenberghe and Boyd and the package `lrs` [**278**] of Avis as subroutines. The interested reader can download the implementation from our web page within our package `rmd` (rigorous MAXDET) [**285**].

With the implemented algorithm we are able to determine the solution of the lattice covering problem and the lattice packing-covering problem in dimensions $d = 1, \dots, 5$. This hereby reproduced the known results. For dimension $d = 5$ this computation took on a 2GHz Intel Pentium computer less than 90 minutes. By our computations we get a conjectural list of approximations of all locally optimal solutions. In Section 5.3.1 we give some details.

More important, with the help of a heuristic method, which we describe in Section 5.3.2, this approach produces interesting new lattices in dimensions $d = 6$ which give better coverings and packing-coverings than previously known ones. We analyze them in Section 5.3.3 and in Section 5.3.4 where we give rigorous proofs of some facts concerning these lattices which we found experimentally. This analysis turned out to be fruitful: The new 6-dimensional lattices were our starting point to find new best-known covering lattices in dimensions 7 and 8. We describe the 7-dimensional case in Section 5.3.5. The 8-dimensional case is explained in Section 5.5.5. Using the extensions of Voronoi's reduction theory described in Chapter 4, we obtain new best known covering lattices in dimension 9–15, as explained in Section 5.3.7 and Section 5.3.8.

**5.3.1. Dimensions** $1, \dots, 5$. We described the solutions to both problems for the 1, 2, and 3-dimensional cases in Section 1.3 and 1.4. Since in these cases there is only one type of Delone triangulation, these results are computationally rather trivial.

The 4-dimensional case requires more work because there are three inequivalent Delone triangulations in dimension 4. Baranovskii [**13**] and independently Dickson [**84**] found all three inequivalent locally optimal solutions to the lattice covering problem in dimension 4. Earlier Delone and Ryshkov showed that the lattice $\mathsf{A}_4^*$ gives the best 4-dimensional lattice covering *without* determining all locally optimal solutions. We describe their approach in Section 5.3.2.

For the lattice packing-covering problem the 4-dimensional case was resolved by Horvath [**139**]. He shows that there exist, as in the covering case, three isolated, locally optimal solutions — one for each Delone triangulation. It is interesting that *Horvath's optimal packing-covering lattice* $\mathsf{Ho}_4$ does not belong to the family of root lattices and their duals. An associated PQF is

$$Q_{\mathsf{Ho}_4} = \begin{pmatrix} 2 & 1 & -1 & -1 \\ 1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} + \frac{1}{3}\sqrt{3} \begin{pmatrix} 4 & 1 & -2 & -2 \\ 1 & 4 & -2 & -2 \\ -2 & -2 & 4 & 1 \\ -2 & -2 & 1 & 4 \end{pmatrix}.$$

The first summand, associated to the best packing lattice $\mathsf{D}_4$, lies on an extreme ray of the secondary cone belonging to $Q_{\mathsf{Ho}_4}$.

In a series of papers Ryshkov and Baranovskii solved the 5-dimensional lattice covering problem. In [**210**] Ryshkov introduced the concept of C-types. Two Delone triangulations are of the same *C-type* if their 1-skeletons (the graph consisting of vertices and edges of the triangulation) coincide. He gave an algorithm to find all inequivalent C-types in any given dimension. He computed that there are 3 inequivalent C-types in dimension 4 and that there are 76 inequivalent C-types in dimension 5. Using this list Baranovskii and Ryshkov enumerated 221 (of 222) inequivalent 5-dimensional Delone triangulations in [**21**]. They described the triangulations in more detail in [**22**]. In the last paper of the series [**215**] they showed that the lattice $\mathsf{A}_5^*$ provides the least dense 5-dimensional lattice covering. In their proof they do not find all locally optimal lattice coverings. By using estimations (the method explained in Section 5.3.2 is one of their main tools) they merely show that all local minima exceed the covering density of $\mathsf{A}_5^*$. Note that, as mentioned in Section 4.1.3, Ryshkov and Baranovskii missed one Delone triangulation. Fortunately, this does not give a thinner lattice covering than $\mathsf{A}_5^*$.

Using our algorithm, and the techniques described in Section 5.2 to obtain certificates, we produced a list of inequivalent locally optimal lattice coverings in dimension 5. This list is conjecturally complete and gives approximations of the local optima. The computation, including all certificates, takes about 90 minutes on a 2GHz Intel Pentium computer. We can prove the following theorem rigorously.

THEOREM 5.24. *In dimension* 5*, there exist at least* 216 *and at most* 218 *inequivalent, local minima of the lattice covering density function* $\Theta$*, ranging from approximately* 2.124286 *to approximately* 2.757993*. All of them are attained in the interior of their secondary cones.*

PROOF. We take a complete list of 222 inequivalent secondary cones, generated by `scc` (see Section 4.1.3 and [**286**]). For each cone, we applied the program `coop` (see Section 5.1.5 and [**285**]). It computes certified bounds for $\Theta$ within the closure of a given secondary cone by solving the determinant maximization problem described in Section 5.1.4.1. The bounds for the duality gap described in Section 5.1.2 are mathematical rigorous, since we use rational arithmetic only. Solving additional determinant maximization problems as described in Section 5.2.2, the program then tries to obtain certificates for the approximated local optimum to be attained in the interior or on the boundary of the secondary cone. As a result, there are 216 certified local optima attained in the interior of secondary cones and 4 certified non-optima attained on the boundary. This leaves two cases which might be local optima, attained in the interior of their secondary cones. However, if they are attained on the boundary (and the numerical evidence strongly supports this) then they are not local optima, because the corresponding secondary cones adjacent to the facets in question contain PQFs with smaller covering density. □

Note that our computations prove rigorously that the PQF of Barnes and Trenerry [**28**] yields the second best locally optimal lattice covering with density of approximately 2.230117.

In 1986, Horvath [**140**] solved the lattice packing-covering in dimension 5. The proof is about 70 pages long (private communication). As in dimension 4, Horvath's lattice $\mathsf{Ho}_5$ is not among previously known ones. An associated PQF is

$$Q_{\mathsf{Ho}_5} = \begin{pmatrix} 2 & 1 & 0 & -1 & -1 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} + \frac{1+\sqrt{13}}{6} \begin{pmatrix} 6 & 3 & -2 & -2 & -2 \\ 3 & 6 & -2 & -3 & -2 \\ -2 & -2 & 6 & -1 & -1 \\ -2 & -3 & -1 & 6 & 0 \\ -2 & -2 & -1 & 0 & 6 \end{pmatrix}.$$

Again, the first summand is associated to the best packing lattice $\mathsf{D}_4$. It has rank 4 and lies on an extreme ray of the secondary cone belonging to $Q_{\mathsf{Ho}_5}$.

By applying propositions of Section 5.2 we were able to reproduce Horvath's result and moreover to attain a list of locally optimal solutions. Again this list is conjecturally complete and gives approximations of the local optima. Compared to the covering problem, there are many more secondary cones that contain no locally optimal solution. The computation, including all certificates, also takes about 90 minutes on a 2GHz Intel Pentium computer. We can prove the following theorem rigorously.

THEOREM 5.25. *In dimension* 5*, there exist at least* 47 *and at most* 75 *local minima of the lattice packing-covering constant* $\gamma$*, ranging from approximately* 1.449456 *to approximately* 1.557564*. At least* 45 *of them are attained in the interior of their secondary cones.*

PROOF. As for the proof of Theorem 5.24, we used the complete list of 222 inequivalent secondary cones, generated by `scc`. For each cone, we applied the program `pacoop` (see Section 5.1.5). It computes certified bounds for $\gamma$ within the closure of a given secondary cone by rigorously solving the semidefinite program described in Section 5.1.4.2. Again the program tries to obtain certificates for the approximated local optimum to be attained in the interior or on the boundary of the secondary cone. As a result, there are 45 certified local optima attained in the interior of secondary cones and 147 certified non-optima attained on the boundary. There are 30 of 222 remaining cases in which the program `pacoop` did not give a certificate for the optimum to be attained in the interior or the boundary of the secondary cone. However, if all of them are attained on the boundary (and the numerical evidence strongly supports this), there are two local optima on the common boundary of three inequivalent secondary cones each. Thus there exist at least 47 local optima as claimed. □

**5.3.2. Heuristic methods.** Before going to dimension 6, let us explain one important heuristic method, which is essential for finding the new lattices. One of the biggest problem in finding good lattice coverings or packing-coverings in dimension 6 and higher is that it is not a priori clear which Delone triangulations admit good ones. Solving a determinant maximization problem is a rather time consuming task and there are a lot of inequivalent Delone triangulations to consider. So a desirable tool is a fast computable lower bound, as described in Section 5.2.6.

We can view the set of Delone triangulations as an undirected labeled graph. A node represents a Delone triangulation and two nodes are adjacent if their Delone triangulations are bistellar neighbors. Let $\mathcal{D}$ be a Delone triangulation. We label its node by local lower bounds $\Theta_*(\mathcal{D})$ or $\gamma_*(\mathcal{D})$, obtained from Propositions 5.21 and 5.23 using a complete list of inequivalent Delone simplices of $\mathcal{D}$. We can use the labeling in two different ways.

On the one hand it is clear that if the labeling of a node is large, then the considered Delone triangulation does not admit a good lattice covering and lattice packing-covering respectively. Delone and Ryshkov [**78**] solved the lattice covering problem in dimension 4 by this method. However, the five-dimensional lattice covering problem cannot be solved in this way. From the 222 inequivalent Delone triangulations there are 20 whose local lower bound is smaller than $\Theta(\mathsf{A}_5^*)$.

On the other hand we can hope that $\mathcal{D}$ admits a good lattice covering or packing-covering if the local lower bound is small. In dimension 6 the hope that good local lower bounds yield good lattice coverings is partially fulfilled. For a detailed account we refer to [**254**] and [**229**].

**5.3.3. New six-dimensional lattices.** We have not been able to solve the lattice covering problem in this dimension. However we found some new interesting covering lattices. As mentioned in Section 1.3, Ryshkov [**208**] asked what is the first dimension $d$ where $\mathsf{A}_d^*$ does not give the least dense lattice covering. Using the heuristic method described in the previous section, we found about 100 inequivalent secondary cones containing lattice coverings better than $\mathsf{A}_6^*$. Thus the answer to Ryshkov's question is that $d = 6$ is the smallest dimension where the lattice $\mathsf{A}_d^*$ does not give the least dense lattice covering.

Two of the new lattice coverings were strikingly good. By the computational optimization we first only obtain numerical estimates of these coverings.

The currently second best-known lattice covering is quite easy to find and comes out of the heuristic method described above in most of the trials. An approximation is given by the PQF

$$Q_6^{c2} \approx \begin{pmatrix} 1.9982 & 0.5270 & -0.4170 & -0.5270 & 0.5270 & -1.0541 \\ 0.5270 & 1.9982 & -0.4170 & -0.5270 & 0.5270 & -1.0541 \\ -0.4170 & -0.4170 & 2.1082 & -1.0541 & -0.4170 & 0.8341 \\ -0.5270 & -0.5270 & -1.0541 & 1.9982 & -0.5270 & -0.4170 \\ 0.5270 & 0.5270 & -0.4170 & -0.5270 & 1.9982 & -1.0541 \\ -1.0541 & -1.0541 & 0.8341 & -0.4170 & -1.0541 & 2.1082 \end{pmatrix}$$

and its covering density is $\Theta(Q_6^{c2}) \approx 2.466125$. The Delone subdivision is a triangulation and its secondary cone has 130 facets. The local lower bound for the Delone subdivision is approximately 2.318034.

The currently best known lattice covering in dimension 6 is given by

$$Q_6^{c1} \approx \begin{pmatrix} 2.0550 & -0.9424 & 1.1126 & 0.2747 & -0.9424 & -0.6153 \\ -0.9424 & 1.9227 & -0.5773 & -0.7681 & 0.3651 & -0.3651 \\ 1.1126 & -0.5773 & 2.0930 & -0.4934 & -0.5773 & -0.9804 \\ 0.2747 & -0.7681 & -0.4934 & 1.7550 & -0.7681 & 0.7681 \\ -0.9424 & 0.3651 & -0.5773 & -0.7681 & 1.9227 & -0.3651 \\ -0.6153 & -0.3651 & -0.9804 & 0.7681 & -0.3651 & 1.9227 \end{pmatrix}$$

with covering density $\Theta(Q_6^{c1}) \approx 2.464802$. The Delone subdivision is a triangulation and its secondary cone has 100 facets. The local lower bound is approximately 2.322204.

Concerning the packing-covering problem, we found in the secondary cone of $Q_6^{c2}$, the PQF

$$Q_6^{pc} \approx \begin{pmatrix} 2.0088 & 0.5154 & 0.5154 & -0.5154 & 0.9778 & 0.5154 \\ 0.5154 & 2.0088 & 0.5154 & -0.5154 & -0.5154 & -0.9778 \\ 0.5154 & 0.5154 & 2.0088 & -0.5154 & -0.5154 & 0.5154 \\ -0.5154 & -0.5154 & -0.5154 & 2.0088 & -0.9778 & -0.5154 \\ 0.9778 & -0.5154 & -0.5154 & -0.9778 & 2.0088 & 0.9778 \\ 0.5154 & -0.9778 & 0.5154 & -0.5154 & 0.9778 & 2.0088 \end{pmatrix},$$

which gives currently the best known lattice packing-covering in dimension 6 with packing-covering constant $\gamma(Q_6^{pc}) \approx 1.411081$.

**5.3.4. Beautification and a unified view.** In this section we examine the new PQFs in greater detail. Although we found an answer to Ryshkov's question, these results are not fully satisfying. We want to know the exact lattices and prove rigorously that they have a good covering density and that they are locally optimal. Even more important, we want to know an interpretation of why these lattice coverings are good. To accomplish this we collect some more data.

The automorphism group of $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2})$ has order 3840 and the automorphism group of $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})$ has order 240. With the knowledge of the groups it is possible to compute the extreme rays of both secondary cones (see [**88**]). It turns out that the secondary cone $\mathbf{C}_1 = \overline{\mathbf{\Delta}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2}))}$ has $7,145,429$ and the cone $\mathbf{C}_2 = \overline{\mathbf{\Delta}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1}))}$ has $2,257,616$ extreme rays. Both contain an extreme ray associated to the lattice $\mathsf{E}_6^*$ given for example by the PQF

$$Q_{\mathsf{E}_6^*} = \begin{pmatrix} 4 & 1 & 2 & 2 & -1 & 1 \\ 1 & 4 & 2 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 & 1 & 2 \\ -1 & 2 & 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 \end{pmatrix}.$$

After transforming $Q_6^{c1}$ and $Q_6^{c2}$ by integral unimodular transformations we can assume that the Delone triangulations of the two PQFs are refinements of the Delone subdivision $\mathrm{Del}_{\mathbb{Z}^d}(Q_{\mathsf{E}_6^*})$. This Delone subdivision was investigated in different contexts (see [**269**], [**66**], [**184**] and [**17**]). We briefly review the main results:

PROPOSITION 5.26. *In the Delone subdivision of $Q_{\mathsf{E}_6^*}$ are in a star (for example of the origin)* 720 *full-dimensional* 6*-dimensional Delone polytopes, and the* *automorphism group of $Q_{\mathsf{E}_6^*}$ acts transitively on them. Each polytope is the convex hull of three regular triangles lying in three pairwise orthogonal affine planes. Each polytope has 9 vertices, 27 facets, and three different triangulations, where each triangulation consists of nine 6-dimensional simplices. The covering density is $\Theta(Q_{\mathsf{E}_6^*}) = \frac{8}{9\sqrt{3}} \cdot \mathrm{vol}(B^6) \approx 2.652071$.*

PROOF. Except for the possible refining triangulations, all this data is well-known, see for example [**66**], *Summary for* $\mathsf{E}_6^*$.

To describe the triangulations we introduce coordinates. A Delone polytope of $Q_{\mathsf{E}_6^*}$ is similar to the polytope $P = \mathrm{conv}\{\boldsymbol{u}_1, \boldsymbol{u}_\omega, \boldsymbol{u}_{\overline{\omega}}, \boldsymbol{v}_1, \boldsymbol{v}_\omega, \boldsymbol{v}_{\overline{\omega}}, \boldsymbol{w}_1, \boldsymbol{w}_\omega, \boldsymbol{w}_{\overline{\omega}}, \}$ where $\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{u}_\omega = -\frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2$, $\boldsymbol{u}_{\overline{\omega}} = -\frac{1}{2}\boldsymbol{e}_1 - \frac{\sqrt{3}}{2}\boldsymbol{e}_2$, $\boldsymbol{v}_1 = \boldsymbol{e}_3$, $\boldsymbol{v}_\omega = -\frac{1}{2}\boldsymbol{e}_3 + \frac{\sqrt{3}}{2}\boldsymbol{e}_4$, $\boldsymbol{v}_{\overline{\omega}} = -\frac{1}{2}\boldsymbol{e}_3 - \frac{\sqrt{3}}{2}\boldsymbol{e}_4$, $\boldsymbol{w}_1 = \boldsymbol{e}_5$, $\boldsymbol{w}_\omega = -\frac{1}{2}\boldsymbol{e}_5 + \frac{\sqrt{3}}{2}\boldsymbol{e}_6$, $\boldsymbol{w}_{\overline{\omega}} = -\frac{1}{2}\boldsymbol{e}_5 - \frac{\sqrt{3}}{2}\boldsymbol{e}_6$. The three possible triangulations are given by the set of nine 6-dimensional simplices

$$\begin{aligned}
\mathcal{T}_{\boldsymbol{u}} &= \{\mathrm{conv}\,\mathrm{vert}\,P \setminus \{\boldsymbol{v}_z, \boldsymbol{w}_{z'}\} : z, z' \in \{1, \omega, \overline{\omega}\}\},\\
\mathcal{T}_{\boldsymbol{v}} &= \{\mathrm{conv}\,\mathrm{vert}\,P \setminus \{\boldsymbol{u}_z, \boldsymbol{w}_{z'}\} : z, z' \in \{1, \omega, \overline{\omega}\}\},\\
\mathcal{T}_{\boldsymbol{w}} &= \{\mathrm{conv}\,\mathrm{vert}\,P \setminus \{\boldsymbol{u}_z, \boldsymbol{v}_{z'}\} : z, z' \in \{1, \omega, \overline{\omega}\}\}.
\end{aligned}$$

To finish the proof one has to show that these sets indeed define triangulations and that they are the only possible triangulations. This can be done by a straightforward computation using the facts that the minimal affine dependent subsets of $\mathrm{vert}\,P$ are

$$\{\boldsymbol{v}_1, \boldsymbol{v}_\omega, \boldsymbol{v}_{\overline{\omega}}, \boldsymbol{w}_1, \boldsymbol{w}_\omega, \boldsymbol{w}_{\overline{\omega}}\}, \{\boldsymbol{u}_1, \boldsymbol{u}_\omega, \boldsymbol{u}_{\overline{\omega}}, \boldsymbol{w}_1, \boldsymbol{w}_\omega, \boldsymbol{w}_{\overline{\omega}}\}, \{\boldsymbol{u}_1, \boldsymbol{u}_\omega, \boldsymbol{u}_{\overline{\omega}}, \boldsymbol{v}_1, \boldsymbol{v}_\omega, \boldsymbol{v}_{\overline{\omega}}\},$$

that the facets of $P$ are $\mathrm{conv}\,\mathrm{vert}\,P \setminus \{\boldsymbol{u}_z, \boldsymbol{v}_{z'}, \boldsymbol{w}_{z''}\}$ where $z, z', z'' \in \{1, \omega, \overline{\omega}\}$, and by applying the following proposition.

PROPOSITION 5.27. *([**200**, Prop. 2.2]) Let $\mathcal{A} \subseteq \mathbb{R}^d$ be a finite point set, and let $\mathcal{T}$ be a set of $d$-dimensional simplices with vertices in $\mathcal{A}$. The set $\mathcal{T}$ defines a triangulation of the polytope $\mathrm{conv}\,\mathcal{A}$ if and only if the following two conditions hold:*

1. *For all $S, S' \in \mathcal{T}$ there exists a minimal affine dependency $\sum_{\boldsymbol{a}\in\mathcal{A}} \lambda_{\boldsymbol{a}} \boldsymbol{a} = \boldsymbol{0}$ with $\sum_{\boldsymbol{a}\in\mathcal{A}} \lambda_{\boldsymbol{a}} = 0$ so that $\{\boldsymbol{a} \in \mathcal{A} : \lambda_{\boldsymbol{a}} > 0\} \subseteq S$ and $\{\boldsymbol{a} \in \mathcal{A} : \lambda_{\boldsymbol{a}} < 0\} \subseteq S'$.*
2. *For all $S \in \mathcal{T}$ and for every $(d-1)$-dimensional facet $F$ of $S$ there exists either a $(d-1)$-dimensional facet $F'$ of $\mathrm{conv}\,\mathcal{A}$ with $F \subseteq F'$ or there exists another simplex $S' \in \mathcal{T} \setminus \{S\}$ also having $F$ as a facet.*

□

Using this information we are able to prove that the PQF $Q_6^{\mathrm{c}1}$ is closely related to $Q_{\mathsf{E}_6^*}$.

THEOREM 5.28. *The PQF $Q_6^{\mathrm{c}1}$ gives the least dense lattice covering among all PQFs whose Delone subdivision refines the Delone subdivision $\mathrm{Del}_{\mathbb{Z}^d}(Q_{\mathsf{E}_6^*})$.*

PROOF. Our proof is computational and uses a branch and cut method. We have to show that all secondary cones of a Delone triangulation refining $\mathrm{Del}_{\mathbb{Z}^d}(Q_{\mathsf{E}_6^*})$ do not contain a PQF with covering density less than $\Theta(Q_6^{\mathrm{c}1})$. There are 40 full-dimensional Delone polytopes of $Q_{\mathsf{E}_6^*}$ which cannot be transformed into each other by translations or by the map $\boldsymbol{x} \mapsto -\boldsymbol{x}$. Since each of these Delone polytopes has 3 possible triangulations, the number of all periodic triangulations refining $\mathrm{Del}_{\mathbb{Z}^d}(Q_{\mathsf{E}_6^*})$ is $3^{40}$. It is not a priori clear how to distinguish between Delone and non-Delone triangulations, and it is not possible to generate all $3^{40}$ triangulations. We choose a backtracking approach instead.

We arrange partial triangulations in a tree. On every level one of the 40 Delone polytopes is triangulated so that we have $3^n$ nodes on the $n$-th level. For every node, respectively partial triangulation $N$ we define the value

$$\max\left\{\det Q : Q \in \mathcal{S}^6_{>0},\ Q \in \mathcal{B}_{L,\leq 1} \text{ for all simplices } L \text{ of } N\right\}.$$

Obviously, from this maximum, we obtain a lower bound $\Theta_N$ for the covering density of any PQF whose Delone subdivision refines the partial triangulation $N$. We can compute a lower bound of this value by solving a determinant maximization problem similar to the one in Section 5.1.4.1. Note that this can be done rigorously using rational arithmetic only if we proceed as described in Section 5.1.2. If the lower bound is larger than 2.464802, we can cut the tree at this node, since the covering density of $Q_6^{c1}$ is less than 2.464801 (see Theorem 5.30).

This algorithm visits exactly 432 nodes of depth 40. One of these triangulations equals $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})$. All the others are equivalent to $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})$ because the automorphism group of $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})$, which has order 240, is a subgroup of $\mathrm{Aut}(Q_{\mathsf{E}_6^*})$ and the order of $\mathrm{Aut}(Q_{\mathsf{E}_6^*})$ equals $103680 = 432 \cdot 240$. $\square$

This computational proof takes about two weeks on a 2GHz Intel Pentium computer. The source code `e6d.cc` is available from our web page as part of the package `rmd` [**285**].

The knowledge of the automorphism groups of $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})$ and $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2})$ also enables us to give a unified view on both lattices. We have

$$\mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1})) \subseteq \mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2})) \subseteq \mathrm{Aut}(Q_{\mathsf{E}_6^*}).$$

The automorphism group $\mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2}))$ turns out to be the subgroup of the group $\mathrm{Aut}(Q_{\mathsf{E}_6^*})$ stabilizing the minimal vectors $\pm\boldsymbol{e}_1$, and $\mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1}))$ is the intersection of the two subgroups of $\mathrm{Aut}(Q_{\mathsf{E}_6^*})$ stabilizing the minimal vectors $\pm\boldsymbol{e}_1$ and $\pm\boldsymbol{e}_2$ respectively.

The subspace $\mathbf{I}_1$ of all quadratic forms invariant under $\mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2}))$ is spanned by the PQFs $Q_{\mathsf{E}_6^*}$ and $R_1$ (see below). At the same time, $\mathbf{I}_1 \cap \mathbf{C}_1$ is a cone with extreme rays $Q_{\mathsf{E}_6^*}, R_1$. By Proposition 5.7, $Q_6^{c2}$ has to lie in $\mathrm{cone}\{Q_{\mathsf{E}_6^*}, R_1\}$. The subspace $\mathbf{I}_2$ of all quadratic forms invariant under the group $\mathrm{Aut}(\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c1}))$ is four-dimensional. The cone $\mathbf{I}_2 \cap \mathbf{C}_2$ has six extreme rays $Q_{\mathsf{E}_6^*}, R_2, \ldots, R_6$, where

$$R_1 = \begin{pmatrix} 12 & 3 & 6 & 6 & -3 & 3 \\ 3 & 7 & 4 & 4 & 3 & 2 \\ 6 & 4 & 8 & 3 & 1 & 4 \\ 6 & 4 & 3 & 8 & 1 & 4 \\ -3 & 3 & 1 & 1 & 7 & 3 \\ 3 & 2 & 4 & 4 & 3 & 7 \end{pmatrix},\ R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 2 & 2 \\ 0 & 2 & 0 & 4 & 2 & 2 \\ 0 & 3 & 2 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 3 & 5 \end{pmatrix},\ R_3 = \begin{pmatrix} 6 & 4 & 4 & 4 & 0 & 2 \\ 4 & 11 & 6 & 6 & 5 & 3 \\ 4 & 6 & 8 & 3 & 3 & 4 \\ 4 & 6 & 3 & 8 & 3 & 4 \\ 0 & 5 & 3 & 3 & 7 & 4 \\ 2 & 3 & 4 & 4 & 4 & 7 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 3 & 2 & 2 & 2 & 0 & 1 \\ 2 & 3 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 1 & 1 & 2 \\ 2 & 2 & 1 & 4 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 2 & 2 & 4 \end{pmatrix},\ R_5 = \begin{pmatrix} 7 & 3 & 4 & 4 & -1 & 2 \\ 3 & 17 & 8 & 8 & 9 & 4 \\ 4 & 8 & 12 & 3 & 5 & 6 \\ 4 & 8 & 3 & 12 & 5 & 6 \\ -1 & 9 & 5 & 5 & 13 & 7 \\ 2 & 4 & 6 & 6 & 7 & 12 \end{pmatrix},\ R_6 = \begin{pmatrix} 9 & 6 & 6 & 6 & 0 & 3 \\ 6 & 9 & 6 & 6 & 3 & 3 \\ 6 & 6 & 8 & 4 & 2 & 4 \\ 6 & 6 & 4 & 8 & 2 & 4 \\ 0 & 3 & 2 & 2 & 5 & 3 \\ 3 & 3 & 4 & 4 & 3 & 6 \end{pmatrix}.$$

Note that $R_2$ lies in $\mathbf{I}_1$. Altogether, this yields the "picture" in dimension 21 given in Figure 2. We refer to [**119**] for another detailed study of Delone subdivisions related to $\mathsf{E}_6^*$.

Let us finally try to find the exact coordinates of the PQFs. This is easy for $Q_6^{c2}$. We know that we can scale $Q_6^{c2}$ so that $Q_6^{c2} = Q_{\mathsf{E}_6^*} + xR_1$, for some $x \in \mathbb{R}_{\geq 0}$. Now, the exact finding of $x$ boils down to finding roots of an univariate polynomial:

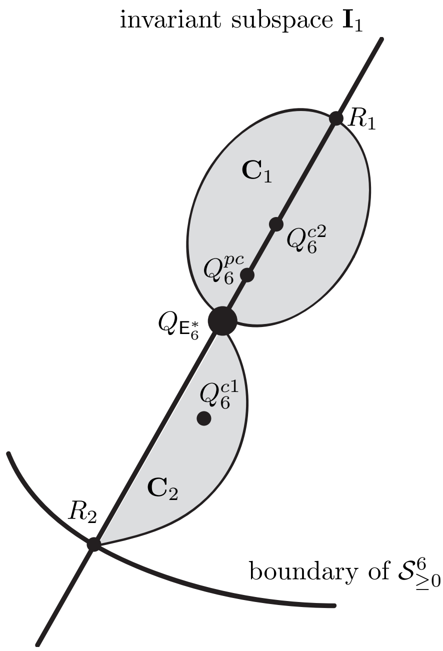


FIGURE 2. Unified view on $Q_{\mathsf{E}_6^*}$, $Q_6^{c1}$,$Q_6^{c2}$ and $Q_6^{pc}$.

we have to minimize the function $x \mapsto \mu(Q_{\mathsf{E}_6^*} + xR_1)^d / \det(Q_{\mathsf{E}_6^*} + xR_1)$ where we know, because of the approximate solution, that $\mu(Q_{\mathsf{E}_6^*} + xR_1)$ is a polynomial for all points in a sufficiently small neighborhood of the exact $x$. This leads to the Ansatz

$$Q_6^{c2} = Q_{\mathsf{E}_6^*} + \frac{\sqrt{1057} - 1}{88} R_1.$$

Now we can use the tools we introduced in Section 5.2 to prove rigorously that this Ansatz works.

THEOREM 5.29. *The PQF $Q_6^{c2}$ gives a locally optimal lattice covering. Its covering density is*

$$\Theta(Q_6^{c2}) = \frac{\sqrt{1124895337698\sqrt{1057} - 33713139497730}}{3543122} \cdot \mathrm{vol}(B^6) \approx 2.466121650.$$

PROOF. Our proof is again computational. We provide the MAGMA program `check_q6c2.m` at the `arXiv.org` e-print archive. To access it, download the source files for the paper `math.MG/0403272`. Since it uses rational arithmetic only, the proof is rigorous.

Let us describe the steps. First we compute the Delone subdivision of $Q_6^{c2}$: We compute the Delone subdivisions of the three PQFs $Q_1 = \frac{9}{10} Q_{\mathsf{E}_6^*} + \frac{3}{10} R_1$, $Q_2 = \frac{11}{10} Q_{\mathsf{E}_6^*} + \frac{3}{10} R_1$, and $Q_3 = \frac{11}{10} Q_{\mathsf{E}_6^*} + \frac{5}{10} R_1$. Then we check that they coincide and that it turns out to be a triangulation. Then we show that $Q_6^{c2} \in \mathrm{conv}\{Q_1, Q_2, Q_3\}$ so that $\mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2}) = \mathrm{Del}_{\mathbb{Z}^d}(Q_1)$. Now we can compute the circumradii of all the Delone simplices by formula (5.7) in the proof of Proposition 5.1. This gives the value $\Theta(Q_6^{c2})$. Finally we prove that $Q_6^{c2}$ gives a locally optimal lattice covering using the criterion of Proposition 5.14. We compute the gradients $g_{L,\mu(Q_6^{c2})}$ for all simplices $L \in \mathrm{Del}_{\mathbb{Z}^d}(Q_6^{c2})$ achieving maximum circumradius. Summing them up yields a multiple of $-(Q_6^{c2})^{-1}$. □

This computational proof takes about one minute on a 2GHz Intel Pentium computer.

What is the general pattern behind the beautification process? Let $Q$ be a locally optimal lattice covering with Delone triangulation $\mathcal{D}$. We use the symmetry of $\mathcal{D}$ to find the subspace in which $Q$ lies. This reduces the number of involved variables. The simplices of the Delone triangulation which have circumradius 1 give equality constraints. Then we maximize the determinant of the quadratic forms lying in the subspace subject to the equality constraints. For this optimization problem, which involves only algebraic equations, we can use Gröbner basis techniques.

Unfortunately, we were not able to solve the corresponding algebraic equations for $Q_6^{c1}$ so we have to be satisfied with an approximation.

THEOREM 5.30. *The covering density of the PQF $Q_6^{c1}$ is at most* 2.464801.

PROOF. The covering density of the PQF $Q_{\mathsf{E}_6^*} + xR_2 + yR_3 + zR_4$ with $x = 0.15266328480099$, $y = 0.32884740614948$, $z = 0.13827491241153$ is smaller than 2.464801. For a computational proof of this fact we provide the MAGMA program `check_q6c1.m`, available from the source files of the paper `math.MG/0403272` at the `arXiv.org` e-print archive. The steps are similar to the first steps of the proof of Theorem 5.29. □

On the basis of Theorem 5.28 and our extensive computational experiments we make the following conjecture.

CONJECTURE 5.31. *The PQF $Q_6^{c1}$ provides the unique least dense lattice covering in dimension* 6.

Using a similar and more successful beautification process for $Q_6^{pc}$, we make the Ansatz

$$Q_6^{pc} = Q_{\mathsf{E}_6^*} + \frac{\sqrt{798} - 18}{79} R_1.$$

THEOREM 5.32. *$Q_6^{pc}$ is a locally optimal solution to the lattice packing-covering problem, lying in the interior of its secondary cone. Its lattice packing covering constant is*

$$\gamma(Q_6^{pc}) = 2\sqrt{2\sqrt{798} - 56} \approx 1.411081242.$$

PROOF. This is similar to the proof of Theorem 5.29. We provide a MAGMA program `check_q6pc.m`, available from the `arXiv.org` e-print archive with the source files of the paper `math.MG/0403272`. □

**5.3.5. Dimension 7.** After analyzing the 6-dimensional case, we got a feeling of where we have to search for good 7-dimensional lattice coverings. We took $Q_{\mathsf{E}_7^*}$ and a lattice vector which is a longest vector of the shortest vectors in the cosets $\mathbb{Z}^d/2\mathbb{Z}^d$. We computed the stabilizer group of this vector and the invariant subspace of this group. By perturbing $Q_{\mathsf{E}_7^*}$ in this subspace randomly, we found a PQF whose Delone subdivision is a triangulation. We solved the determinant maximization problem of Section 5.1.4.1 which belongs to this Delone triangulation and found the PQF

$$Q_7^c = \begin{pmatrix} 12 & 1 & 1 & 1 & 1 & 1 & 5 \\ 1 & 12 & 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & 12 & 1 & 1 & 1 & 5 \\ 1 & 1 & 1 & 12 & 1 & 1 & 5 \\ 1 & 1 & 1 & 1 & 12 & 1 & 5 \\ 1 & 1 & 1 & 1 & 1 & 12 & -6 \\ 5 & 5 & 5 & 5 & 5 & -6 & 14 \end{pmatrix}.$$

We are quite surprised that this PQF has rational entries and was not known before.

THEOREM 5.33. *$Q_7^c$ is a locally optimal solution to the lattice covering problem, lying in the interior of its secondary cone. Its inhomogeneous minimum is $\mu = \frac{15}{2}$, its determinant is* $\det Q_7^c = 2 \cdot 11^6$, *so that* $\Theta(Q_7^c) \approx 2.900024$.

PROOF. Again, this is similar to the proof of Theorem 5.29. We provide the MAGMA program `check_q7c.m`, available from the source files of the paper `math.MG/0403272` at the `arXiv.org` e-print archive. □

**5.3.6. Coxeter lattices.** A new sphere covering in dimension 8 is described in Section 5.5.6. The new sphere coverings in dimension $d \geq 9$ are either *Coxeter lattices* $\mathsf{A}_d^r$ or closely related to them. The lattices $\mathsf{A}_d^r$ are defined whenever $d \geq 2$ and $r$ devides $d+1$ (see [**72**]). A possible definition is via the root lattice

$$\mathsf{A}_d = \{\boldsymbol{x} \in \mathbb{Z}^{d+1} : \sum_{i=0}^{d+1} x_i = 0\}.$$

The Coxeter lattice is the lattice generated by $\mathsf{A}_d$ and the vector

$$(1/r)\left(\sum_{i=1}^{d+1} \boldsymbol{e}_i\right) - (\boldsymbol{e}_1 + .... + \boldsymbol{e}_{(d+1)/r}),$$

where $\boldsymbol{e}_i$ denotes the $i$-th standard basis vector. It is the unique sublattice of $\mathsf{A}_d^*$ containing $\mathsf{A}_d$ as a sublattice of index $r$. In particular $\mathsf{A}_d^* = \mathsf{A}_d^{d+1}$. Other well known lattices in the series are $\mathsf{A}_7^2 = \mathsf{E}_7$, $\mathsf{A}_7^4 = \mathsf{E}_7^*$ and $\mathsf{A}_8^3 = \mathsf{E}_8$.

Since the symmetric group $\mathsf{S}_{d+1}$ acts on the lattice $\mathsf{A}_d^r$, it is possible (with the help of a computer) to enumerate all its orbits of Delone polytopes and to compute their covering densities in fairly large dimensions. A detailed description of the method together with results up to dimension 27 can be found in [**92**].

Baranovskii [**18**] computed (by hand) the Delone decomposition of $\mathsf{A}_9^5$, finding the former best known sphere covering in dimension 9. Anzin [**2**] computed the Delone decomposition of $\mathsf{A}_{11}^4$ and $\mathsf{A}_{13}^7$ establishing the former covering records in those dimensions. In a private communication he reported on computing the covering densities of $\mathsf{A}_{14}^5$ and $\mathsf{A}_{15}^8$, which were also best known ones in their dimension. Hence suitable Coxeter lattices provide good covering lattices. Nevertheless, by applying our theory, we found that all of the five mentioned lattices do not even give a locally optimal lattice covering. However, these lattices give good starting points. In fact, we obtained the new covering records in dimensions $d = 9, 11, 13, 14, 15$ by applying our theory to a suitable linear subspace $T$ containing a PQF of the corresponding Coxeter lattices. More details are given in the next section. For dimensions $d = 10, 12$ we used a lamination technique, which is described thereafter.

**5.3.7. Large subgroups and small linear subspaces.** One strategy to find good covering lattices is to consider subspaces $T$ containing a PQF, which gives a good or even best known sphere covering. In order to keep the number of $T$-secondary cones low (manageable), one preferably chooses a subspace $T$ of low dimension, e.g. less than or equal to 4. Another problem is the size of Delone subdivisions to be dealt with. If $T$ is contained in a space of invariant forms $\mathcal{F}(G)$ of a preferably large finite subgroup $G$ of $\mathsf{GL}_d(\mathbb{Z})$, then this problem can be reduced by exploiting the symmetries of the Delone subdivisions, respectively those of the forms $Q \in \mathcal{F}(G)$ (see Proposition 5.4).

**Dimension 9**. We optimized over a 3-dimensional subspace $T$ containing a PQF of the former record lattice $\mathsf{A}_9^5$. We computed 210 $T$-generic $T$-secondary cones. The new covering record is attained by a PQF which has 34 orbits of Delone polytopes.

**Dimension 11**. We optimized over a 3-dimensional subspace $T$ containing a PQF of the former record lattice $\mathsf{A}_{11}^4$. We computed 2444 $T$-generic $T$-secondary cones. The new covering record is attained by a PQF which has 99 orbits of Delone polytopes.

**Dimension 13**. We optimized over a 2-dimensional subspace $T$ containing a PQF of the former record lattice $\mathsf{A}_{13}^7$. We computed 79 $T$-generic $T$-secondary cones. The new covering record is attained by a PQF which has 134 orbits of Delone polytopes.

**Dimension 14**. We optimized over a 2-dimensional subspace $T$ containing a PQF of the former record lattice $\mathsf{A}_{14}^5$. We computed 162 $T$-generic $T$-secondary cones. The new covering record is attained by a PQF which has 983 orbits of Delone polytopes.

**Dimension 15**. We optimized over a 2-dimensional subspace $T$ containing a PQF of the former record lattice $\mathsf{A}_{15}^8$. We computed 109 $T$-generic $T$-secondary cones. The new covering record is attained by a PQF which has 203 orbits of Delone polytopes.

**5.3.8. Best coverings from laminations.** Another fruitful strategy is the construction of good coverings from lower dimensional ones. Assume that $Q \in \mathcal{S}^d_{>0}$ is contained in a linear subspace $T$ of $\mathcal{S}^d$. We consider a corresponding lattice $L = A\mathbb{Z}^d$ with $A \in \mathsf{GL}_d(\mathbb{R})$. Further, we choose a point $A\boldsymbol{c}$ with $\boldsymbol{c} \in \mathbb{R}^d$ and consider all lattices $L_\lambda$ in $\mathbb{R}^{d+1}$ generated by

$$A' = \begin{pmatrix} A & A\boldsymbol{c} \\ 0 & \lambda \end{pmatrix} \in \mathsf{GL}_{d+1}(\mathbb{R})$$

with $\lambda > 0$ and associated PQF

$$Q' = \begin{pmatrix} Q & Q\boldsymbol{c} \\ \boldsymbol{c}^t Q & \boldsymbol{c}^t Q\boldsymbol{c} + \lambda^2 \end{pmatrix} \in \mathcal{S}^{d+1}_{>0}.$$

Choosing $Q$ within $T$ and $\lambda^2 \in \mathbb{R}$, we obtain a linear subspace $T'$ of forms $Q'$ with $\dim T' = \dim T + 1$. If $A\boldsymbol{c}$ is the center of a Delone polytope $P$ of the lattice $L$, then the lattice vectors in the lattice $L_\lambda$ belonging to the $i$-th layer

$$\left\{ \begin{pmatrix} A & A\boldsymbol{c} \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ i \end{pmatrix} : \boldsymbol{v} \in \mathbb{Z}^d \right\}, \quad \text{with } i \in \mathbb{Z},$$

have at least distance $|i|\lambda$ from $(A\boldsymbol{c}, 0)$. Hence, $P$ is a Delone polytope embedded in $L_\lambda$ when $\lambda$ is at least the circumradius of $P$ in $L$.

**Dimension 10**. We considered the 1-dimensional linear space $T$ containing $\mathsf{A}_9^5$. We look at the linear subspace $T'$ of $\mathcal{S}^{10}$ of dimension 2 constructed with the center of a Delone polytope having the largest symmetry group in $\mathsf{A}_9^5$. We computed 6 $T'$-generic $T'$-secondary cones. The new covering record is attained by a PQF which has 4 orbits of Delone polytopes.

**Dimension 12**. We considered the 1-dimensional linear space $T$ containing $\mathsf{A}_{11}^4$. We look at the linear subspace $T'$ of $\mathcal{S}^{12}$ of dimension 2 constructed with the center of a Delone polytope. We computed 241 $T'$-generic $T'$-secondary cones. The new covering record is attained by a PQF which has 1206 orbits of Delone polytopes. Here we tried all different centers. The new covering record was produced by using a center of a Delone polytope having the third largest symmetry group.

**5.3.9. New best known packing-coverings.** Finally we mention briefly that the discussed approaches for the lattice covering problem, can also be applied to the lattice packing-covering problem. So far, optimizing over suitable $T$-secondary cones with respect to 4-dimensional subspaces $T$ of $\mathcal{S}^7$, we found a new best known 7-dimensional packing-covering lattice with $\gamma(L) = 1.499399\ldots$. The former record holder was the lattice $\mathsf{E}_7^*$ with $\gamma(\mathsf{E}_7^*) = \sqrt{7/3} = 1.527525\ldots$. It remains to undertake a systematic search for further best known, maybe optimal packing-covering lattices.

## 5.4. Classification of totally real thin number fields

In this section we apply our theory to a problem in algebraic number theory. Recently, Bayer–Fluckiger introduced in [**34**] the notion of thin algebraic number fields. A thin algebraic number field is Euclidean due to a special geometric reason, which we explain in Section 5.4.1. Bayer–Fluckiger proved that there exist only finitely many thin number fields (see [**34**, Proposition 11.4]).

In [**35**] Bayer–Fluckiger and Nebe gave a complete list ([**35**, Theorem 5.1]) of 17 candidates for totally real thin algebraic number fields. They proved that 13 of them are thin and one of them is weakly thin but not thin. However, in three cases they did not know how to decide whether the fields are thin or not. Using our theory we finish the classification of totally real thin algebraic number fields. In particular we show in Section 5.4.2 that the three open cases do not give weakly thin fields.

**5.4.1. Background and definitions.** Let us recall some standard definitions from algebraic number theory.

Let $K$ be a number field of degree $n = [K : \mathbb{Q}]$. From now on we assume that $K$ is *totally real*, that is, for all embeddings of fields $\sigma_i : K \to \mathbb{C}$ with $i = 1, \ldots, n$ we have $\sigma_i(K) \subseteq \mathbb{R}$.

Let $\mathfrak{o}_K$ be its ring of integers which is a $\mathbb{Z}$-module of rank $n$. Thus we can write $\mathfrak{o}_K = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. We embed $\mathfrak{o}_K$ into $\mathbb{R}^n$ via the map $\sigma$ defined by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x))$. Hence, $\sigma(\mathfrak{o}_K)$ is a lattice of rank $n$.

By $\mathcal{P}$ we denote the set of $\alpha \in K$ with $\sigma_i(\alpha) > 0$ for all $i = 1, \ldots, n$. Then

$$\langle x, y\rangle_\alpha = \sum_{i=1}^{n} \sigma_i(\alpha x y) = \sum_{i=1}^{n} \sigma_i(\alpha)\sigma_i(x)\sigma_i(y)$$

defines an inner product on $K$ and so on $\mathbb{R}^n$. Every inner product $\langle \cdot, \cdot\rangle : K \times K \to \mathbb{R}$ with the additional property $\langle x, yz\rangle = \langle xz, y\rangle$ for all $x, y, z \in K$ is of this form.

We denote by $\Lambda_{K,\alpha}$ the lattice which is given by the pair $(\mathfrak{o}_K, \langle\cdot,\cdot\rangle_\alpha)$ where $\alpha \in \mathcal{P}$. We say that $K$ is *weakly thin* if for $\Theta(K) = \min_{\alpha\in\mathcal{P}} \Theta(\Lambda_{K,\alpha})$ we have the inequality

$$\Theta(K) \leq \sqrt{\frac{n^n}{(\det \Lambda_{K,1})^2}} \cdot \mathrm{vol}(B^n),$$

and we say that $K$ is *thin* if we have strict inequality. By $t(K)$ we denote the number on the right hand side of this inequality.

Let us briefly explain the motivation of this definition. First note that the discriminant $d_K$ of $K$ equals $(\det \Lambda_{K,1})^2$. Let $N : K \to \mathbb{R}$ be the norm of $K$ which is given by $N(x) = \prod_{i=1}^n \sigma_i(x)$. A number field $K$ is called *Euclidean* if its ring of integers $\mathfrak{o}_K$ is a Euclidean ring with respect to the absolute value of the norm function. Equivalently, $K$ is Euclidean if and only if its *Euclidean minimum*

$$M(K) = \sup_{x\in K} \inf_{y\in\mathfrak{o}_K} |N(x-y)|$$

is strictly less than 1. It is an open problem if there exists a finite or an infinite number of Euclidean number fields. In [**167**] Lenstra, motivated by work of Hurwitz, gave several bounds on the Euclidean minimum of a number field using methods from "Geometry of Numbers". In [**34**] Bayer–Fluckiger extended these results. By using the inequality between the arithmetic and geometric means to relate the norm function $N$ to the norm $\|x\|_\alpha = \sqrt{\langle x, x\rangle_\alpha} = \sqrt{\sum_{i=1}^n \sigma_i(\alpha x^2)}$, she showed that thin fields are Euclidean fields (see [**34**, Proposition 11.2]) and that there exist only finitely many thin fields ([**34**, Proposition 11.4]).

**5.4.2. Classification.** By applying the lower bound for sphere coverings of Coxeter, Few and Rogers [**73**], Bayer–Fluckiger showed that the degree of a totally real thin number field is at most 5 (see [**34**, Proof of Proposition 11.4]). All number fields having low degree and low discriminant are known ([**277**]). Using this list Bayer–Fluckiger and Nebe gave a complete list of 17 candidates of totally real number fields which might be thin. In Table 1 we list these candidates $K$ together with the relevant parameters: The degree $n$, the discriminant $d_K$, the bound $t(K)$ we defined above and the minimum covering density $\Theta(K)$.

Bayer–Fluckiger and Nebe showed by giving a number $\alpha$, which defines the appropriate inner product $\langle\cdot,\cdot\rangle_\alpha$, that 13 candidates are thin. For computing an upper bound of $\Theta(K)$ as given in the table we used their values $\alpha \in \mathcal{P}$ in all but one case. In the case of $\mathbb{Q}[x]/(x^3 + x^2 - 3x - 1)$ their table contains a misprint. Instead of $\alpha = 1 - 18\bar{x} + 10\bar{x}^2$ which does not lie in $\mathcal{P}$ we use $\alpha = 2 - \bar{x}$ instead. For one candidate Bayer–Fluckiger and Nebe showed that it is weakly thin. In this case $t(K) = \Theta(K) = \Theta_4$ where $\Theta_4$ is the least lattice covering density among all 4-dimensional lattices. It is uniquely attained by the lattice $\mathsf{A}_4^*$.

The other three cases were left open by Bayer–Fluckiger and Nebe and by our computation it turns out that they do not give thin fields. Generally, for a totally real number field $K$, there exists a subspace $T$ of $\mathcal{S}^n$ whose dimension equals the degree of $K$, and such that $\Theta(K) = \Theta_T$. The corresponding subspace $T$ is given by the basis $\left(\langle\omega_i, \omega_j\rangle_{\alpha_k}\right)_{1\leq i,j\leq n}$, $k = 1,\ldots,n$, where $\mathfrak{o}_K = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ and $\alpha_1,\ldots,\alpha_n$ forms a $\mathbb{Q}$-basis of $K$ with $\alpha_k \in \mathcal{P}$. Thus if there exist only finitely many inequivalent $T$-secondary cones, we can compute $\Theta(K) = \Theta_T$ by the methods explained in Section 5.1.

| $n$ | $d_K$ | $K$ | $t(K)$ | $\Theta(K)$ | thin |
|---|---|---|---|---|---|
| 2 | 5 | $\mathbb{Q}[x]/(x^2-5)$ | $\geq 2.8099$ | $\leq 1.2645$ | yes |
| | 8 | $\mathbb{Q}[x]/(x^2-2)$ | $\geq 2.2214$ | $\leq 1.3463$ | yes |
| | 12 | $\mathbb{Q}[x]/(x^2-3)$ | $\geq 1.8137$ | $\leq 1.2092$ | yes |
| | 13 | $\mathbb{Q}[x]/(x^2-13)$ | $\geq 1.7426$ | $\leq 1.5708$ | yes |
| | 17 | $\mathbb{Q}[x]/(x^2-17)$ | $\geq 1.5238$ | $\leq 1.2497$ | yes |
| | 21 | $\mathbb{Q}[x]/(x^2-21)$ | $\geq 1.3711$ | $\leq 1.2242$ | yes |
| | 24 | $\mathbb{Q}[x]/(x^2-6)$ | $\geq 1.2825$ | $\leq 1.2583$ | yes |
| 3 | 49 | $\mathbb{Q}[x]/(x^3+x^2-2x-1)$ | $\geq 3.1093$ | $\leq 1.5584$ | yes |
| | 81 | $\mathbb{Q}[x]/(x^3-3x+1)$ | $\geq 2.4183$ | $\leq 2.1225$ | yes |
| | 148 | $\mathbb{Q}[x]/(x^3+x^2-3x-1)$ | $\geq 1.7891$ | $\leq 1.7014$ | yes |
| | 169 | $\mathbb{Q}[x]/(x^3+x^2-4x+1)$ | $\leq 1.6743$ | $\geq 1.8544$ | no |
| 4 | 725 | $\mathbb{Q}[x]/(x^4-x^3-3x^2+x+1)$ | $\geq 2.9323$ | $\leq 2.7045$ | yes |
| | 1125 | $\mathbb{Q}[x]/(x^4-x^3-4x^2+4x+1)$ | $\geq 2.3540$ | $\leq 2.2935$ | yes |
| | 1600 | $\mathbb{Q}[x]/(x^4+2x^3-5x^2-6x-1)$ | $\leq 1.9740$ | $\geq 2.2853$ | no |
| | 1957 | $\mathbb{Q}[x]/(x^4-4x^2-x+1)$ | $\leq 1.7849$ | $\geq 1.8939$ | no |
| | 2000 | $\mathbb{Q}[x]/(x^4-5x^2+5)$ | $=\Theta_4$ | $=\Theta_4$ | weakly |
| 5 | 14641 | $\mathbb{Q}[x]/(x^5+x^4-4x^3-3x^2+3x+1)$ | $\geq 2.4318$ | $\leq 2.2961$ | yes |

TABLE 1. Classification of totally real thin number fields.

For the field $K=\mathbb{Q}[x]/(x^3+x^2-4x+1)$ we have 7 generic $T$-secondary cones and for all lattices $\Lambda_{K,\alpha}$ our computation proves the bound $\Theta(\Lambda_{K,\alpha})\geq 1.8544$. For the field $K=\mathbb{Q}[x]/(x^4+2x^3-5x^2-6x-1)$ we have 47 generic $T$-secondary cones and for all lattices $\Lambda_{K,\alpha}$ our computation shows $\Theta(\Lambda_{K,\alpha})\geq 2.2853$. For the field $K=\mathbb{Q}[x]/(x^4-4x^2-x+1)$ we have 341 generic $T$-secondary cones and for all lattices $\Lambda_{K,\alpha}$ our computation gives $\Theta(\Lambda_{K,\alpha})\geq 1.8939$.

The fact that in all these cases the number of inequivalent $T$-generic secondary cones is finite comes as a pleasant surprise. In this situation $T$ is not the space of invariant forms of a Bravais group so that we cannot a priori rely on the finiteness result of Section 4.3.3. It remains an interesting open problem to give a proof of this fact, which does not rely on the complete enumeration of $T$-secondary cones.

## 5.5. The Leech lattice and the root lattice $\mathsf{E}_8$

The Leech lattice is the exceptional lattice in dimension 24. Soon after its discovery by Leech [**164**] it was conjectured that it is extremal for several geometric problems in $\mathbb{R}^{24}$: the kissing number problem, the sphere packing problem and the sphere covering problem.

In 1979, Odlyzko and Sloane and independently Levenshtein solved the kissing number problem in dimension 24 by showing that the Leech lattice gives an optimal solution. Two years later, Bannai and Sloane showed that it gives the unique solution up to isometries (see [**69**, Ch. 13, 14]). Unlike the kissing number problem, the other two problems are still open.

As mentioned in Section 1.2.4, Cohn and Kumar [**63**] recently showed that the Leech lattice gives the unique densest lattice sphere packing in $\mathbb{R}^{24}$. Furthermore they showed, that the density of any sphere packing (without restriction to lattices) in $\mathbb{R}^{24}$ cannot exceed the one given by the Leech lattice by a factor of more than $1+1.65\cdot 10^{-30}$.

At the moment it is not clear how one can prove a corresponding result for the sphere covering problem. Here, we take a first step into this direction by proving local optimality.

### 5.5.1. Local lattice covering optimality.

THEOREM 5.34. *The Leech lattice gives a sphere covering which is locally optimal among lattices.*

As a direct consequence we obtain the local optimality of the Leech lattice for the lattice packing-covering problem.

COROLLARY 5.35. *The Leech lattice gives a sphere packing-covering which is locally optimal among lattices.*

Surprisingly, a result similar to Theorem 5.34 does not hold for the root lattice $\mathsf{E}_8$, which is the exceptional lattice in dimension 8. As the Leech lattice, $\mathsf{E}_8$ gives the unique solution to the kissing number problem (see [**69**, Ch. 13, 14]) and is conjectured to be extremal for the sphere packing problem. As mentioned in Section 1.2.4, Blichfeldt [**43**] showed that it gives a optimal lattice sphere packing. Later, Vetchinkin [**261**] showed that sphere packings of all other 8-dimensional lattices are less dense. Besides giving another proof of this, Cohn and Kumar [**63**] demonstrated that the density of a sphere packing in $\mathbb{R}^8$ cannot exceed the one of $\mathsf{E}_8$ by a factor of more than $1 + 10^{-14}$. In contrast to the Leech lattice, $\mathsf{E}_8$ cannot be an extremal sphere covering in its dimension though. Conway and Sloane note in [**69**, Ch. 2]: “It is surprising that $\mathsf{A}_8^*$, with [covering density] $\Theta = 3.6658\ldots$, is better than $\mathsf{E}_8$, which has $\Theta = 4.0587\ldots$”. Here, we show that $\mathsf{E}_8$ does not even give a locally optimal lattice sphere covering, by constructing a new 8-dimensional lattice, which yields a less dense sphere covering and whose Delone subdivision has a common refinement with the one of $\mathsf{E}_8$.

THEOREM 5.36. *There exists a lattice with covering density* $\Theta < 3.2013$ *whose Delone subdivision has a common refinement with the one of* $\mathsf{E}_8$.

Note that this new sphere covering beats the old record holder $\mathsf{A}_8^*$ in dimension 8. The proof of Theorem 5.36 relies on the computational methods developed in Section 5.1. Up to now we can only give an approximation of the best “known” covering lattice. Nevertheless, by Proposition 5.7 of Barnes and Dickson, Theorem 5.36 yields the following corollary.

COROLLARY 5.37. *The root lattice* $\mathsf{E}_8$ *does not give a locally optimal lattice sphere covering.*

By Theorem 5.34, we give a first example of a locally optimal covering lattice whose Delone subdivision is not simplicial. By this we give an affirmative answer to a question of Dickson posed in [**85**]. The Leech lattice gives even a strongest possible example, in the sense that it is *rigid*, to be explained in Section 5.5.2. Our proof there immediately applies to $\mathsf{E}_8$ as well, giving a new proof of $\mathsf{E}_8$’s rigidity, first observed by Baranovskii and Grishukhin [**20**].

THEOREM 5.38. *The Leech lattice and* $\mathsf{E}_8$ *are rigid.*

This answers in particular a question of Dickson affirmatively. In [**85**], he states: “Whether it is possible for $f_0$ [a PQF giving a locally optimal lattice covering] to occur on the boundary of a cone [...] is still a matter of conjecture.”

**5.5.2. Rigidity of the Leech lattice and the root lattice $\mathsf{E}_8$.** Recall from Section 4.1 (in particular Proposition 4.6) that the secondary cones of Delone subdivisions (with vertices in $\mathbb{Z}^d$) form a polyhedral subdivision of $\mathcal{S}^d_{\geq 0}$. If a face is contained in the boundary of a second face, then the corresponding Delone subdivision of the second is a true refinement of the first one. If the relative interior of a face in $\mathcal{S}^d_{>0}$ is of minimal dimension 1, the PQFs it contains are called *rigid*. For them, any local change which is not a dilation, changes the Delone subdivision.

We transfer the terminology of Delone subdivisions from PQFs to lattices by saying that the Delone subdivision of the lattice $L'$ is a *refinement* of the Delone subdivision of the lattice $L$, if there are associated PQFs $Q'$ and $Q$ so that the Delone subdivision of $Q'$ is a refinement of the Delone subdivision of $Q$. A lattice is called *rigid* if an associated PQF is rigid.

Now let us introduce the two exceptional lattices, to show that they are rigid. We gathered the information mostly from [**69**, Ch. 4 §8,§11].

The Leech lattice $\Lambda$ with associated PQF $Q_\Lambda$ satisfies $\det(Q_\Lambda) = 1$, $\lambda(Q_\Lambda) = 4$ and $\mu(Q_\Lambda) = 2$. Thus, its packing density, already given by Leech [**164**], is $\delta(Q_\Lambda) = \mathrm{vol}(B^{24})$ and best possible among all lattices in $\mathbb{R}^{24}$ (see [**63**, Th. 9.3]). Its covering density is $\Theta(Q_\Lambda) = 4096 \cdot \mathrm{vol}(B^{24})$. The first proof of this fact is due to Conway, Parker and Sloane [**69**, Ch. 23], where they determine the Delone subdivision of the Leech lattice (cf. [**19**] for another approach). There they also classified the 23 different (up to congruences) Delone polytopes of $Q_\Lambda$ attaining the maximum circumradius $\sqrt{2}$.

From their list, we will consider Delone polytopes of type $\mathsf{A}_1^{24}$ to prove the rigidity in this section and those of type $\mathsf{A}_{24}$ to prove the local optimality in Section 5.5.3. To describe them we define $Q_\Lambda(n) = \{\boldsymbol{v} \in \mathbb{Z}^{24} : Q_\Lambda[\boldsymbol{v}] = 2n\}$. The Delone polytopes of type $\mathsf{A}_1^{24}$ are 24-dimensional regular cross polytopes with respect to the metric induced by $Q_\Lambda$. They are of the form $\boldsymbol{v} + \mathrm{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{47}\}$, $\boldsymbol{v} \in \mathbb{Z}^{24}$, where $\boldsymbol{v}_0 = 0$, $\boldsymbol{v}_{24} \in Q_\Lambda(4)$ and all the other $\boldsymbol{v}_i \in Q_\Lambda(2)$ satisfy $\boldsymbol{v}_j + \boldsymbol{v}_{j+24} = \boldsymbol{v}_{24}$, $j = 0, \ldots, 47$ (indices computed modulo 48). The Delone polytopes of type $\mathsf{A}_{24}$ are 24-dimensional simplices having 275 edge vectors in $Q_\Lambda(2)$ and 25 in $Q_\Lambda(3)$. This is the only information about $\mathsf{A}_{24}$ we will need in Section 5.5.3.

For the proof of rigidity it is convenient to work with the following coordinates with respect to the standard basis of $\mathbb{R}^{24}$: The vectors of squared length 4 of $\Lambda$ are of shape $\frac{1}{\sqrt{8}}((\pm 4)^2 0^{22})$, $\frac{1}{\sqrt{8}}((\pm 2)^8 0^{16})$ and $\frac{1}{\sqrt{8}}(\mp 3 (\pm 1)^{23})$, where permitted permutations of coordinates and permitted positions of minus signs are explained in [**69**, Ch. 10 §3.2]. Here, we only need the $2^2\binom{24}{2}$ vectors of the first type, where all permutations of coordinates and all positions of minus signs are allowed.

The root lattice $\mathsf{E}_8$ with associated PQF $Q_{\mathsf{E}_8}$ satisfies $\det(Q_{\mathsf{E}_8}) = 1$, $\lambda(Q_{\mathsf{E}_8}) = 2$ and $\mu(Q_{\mathsf{E}_8}) = 1$. Thus, its packing density is $\delta(Q_{\mathsf{E}_8}) = \frac{1}{16} \cdot \mathrm{vol}(B^8)$ and best possible among all lattices in $\mathbb{R}^8$ ([**43**]). Its covering density is $\Theta(Q_{\mathsf{E}_8}) = \mathrm{vol}(B^8)$. The first proof of this fact is due to Coxeter [**71**], who gave a complete description of the Delone subdivision of $Q_{\mathsf{E}_8}$. It is a tiling of $\mathbb{R}^8$ into regular simplices and regular cross polytopes with respect to the metric induced by $Q_{\mathsf{E}_8}$. Define $Q_{\mathsf{E}_8}(n) = \{\boldsymbol{v} \in \mathbb{Z}^8 : Q_{\mathsf{E}_8}[\boldsymbol{v}] = n\}$. Note that this slightly differs from the definition of $Q_\Lambda(n)$ above. Then the cross polytopes are of the form $\boldsymbol{v} + \mathrm{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{15}\}$, $\boldsymbol{v} \in \mathbb{Z}^8$, where $\boldsymbol{v}_0 = 0$, $\boldsymbol{v}_8 \in Q_{\mathsf{E}_8}(4)$ and all the other $\boldsymbol{v}_i \in Q_{\mathsf{E}_8}(2)$ satisfy $\boldsymbol{v}_j + \boldsymbol{v}_{j+8} = \boldsymbol{v}_8$, $j = 0, \ldots, 15$ (indices computed modulo 16). The regular simplices have 36 edge vectors in $Q_{\mathsf{E}_8}(2)$. We shall give more information about this Delone subdivision in Section 5.5.5.

For the proof of $\mathsf{E}_8$'s rigidity and for the construction of a new sphere covering in Section 5.5.5 it is again convenient to work with explicit coordinates with respect to the standard basis of $\mathbb{R}^8$. Set

$$\mathsf{E}_8 = \{\boldsymbol{x} \in \mathbb{R}^8 : \boldsymbol{x} \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \text{ and } \sum_{i=1}^{8} x_i \in 2\mathbb{Z}\}. \tag{5.13}$$

The automorphism group of $\mathsf{E}_8$ is generated by all permutations of the 8 coordinates, by all even sign changes and by the matrix $H = \operatorname{diag}(H_4, H_4)$ where

$$H_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

There are 240 vectors of squared length 2 in $\mathsf{E}_8$: $2^2\binom{8}{2}$ of shape $((\pm 1)^2 0^6)$ and $2^7$ of shape $((\pm\frac{1}{2})^8)$ where the number of minus signs is even. The 2160 vectors of squared length 4 are: $2 \cdot 8$ of shape $((\pm 2)0^7)$, $2^4\binom{8}{4}$ of shape $((\pm 1)^4 0^4)$ and $2^7 \cdot 8$ of shape $(\pm\frac{3}{2}(\pm\frac{1}{2})^7)$ where the number of minus signs is odd.

Now we proceed to the proof of Theorem 5.38. We will handle both cases simultaneously. For this, denote by $L$ the Leech lattice $\Lambda$ or the root lattice $\mathsf{E}_8$ and write $d$ for the rank of $L$. We shall show that every PQF $Q$, whose Delone subdivision contains the above mentioned cross polytopes, is a multiple of $Q_L$. By $\langle \cdot, \cdot \rangle$ we denote the inner product given by $Q_L$, i.e. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t Q_L \boldsymbol{y}$, and by $(\cdot, \cdot)$ we denote the inner product given by $Q$.

Let $\boldsymbol{v}, \boldsymbol{w} \in Q_L(2)$ with $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$. So, $\boldsymbol{v}+\boldsymbol{w} \in Q_L(4)$. Therefore, $\boldsymbol{0}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{w}$ are vertices of a Delone cross polytope, as considered above. Let $\boldsymbol{c}$ be the center of its circumsphere, hence $Q[\boldsymbol{0} - \boldsymbol{c}] = Q[\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{c}] = Q[\boldsymbol{v} - \boldsymbol{c}] = Q[\boldsymbol{w} - \boldsymbol{c}]$. Then a straightforward calculation reveals $(\boldsymbol{v}, \boldsymbol{w}) = 0$.

Now we switch to coordinates with respect to the standard basis. Choosing a basis $A$ of $L$ gives the associated PQF $Q_L = A^t A$. Obviously, $Q = A^t (A^t)^{-1} Q A^{-1} A$. We denote the entries of $C = (A^t)^{-1} Q A^{-1}$ by $(c_{ij})$ and by $\boldsymbol{e}_i$ we denote the $i$-th canonical basis vector of $\mathbb{R}^d$. Let $\boldsymbol{v}_i \in \mathbb{R}^d$ be the coordinate vector of $\boldsymbol{e}_i$ with respect to the basis $A$, that is $A\boldsymbol{v}_i = \boldsymbol{e}_i$. Then by our choice of coordinates and by the argument above we have for $i \neq j$ the orthogonality

$$\begin{aligned} 0 &= (\boldsymbol{e}_i + \boldsymbol{e}_j)^t (\boldsymbol{e}_i - \boldsymbol{e}_j) = (\boldsymbol{v}_i + \boldsymbol{v}_j)^t Q_L (\boldsymbol{v}_i - \boldsymbol{v}_j) \\ &= (\boldsymbol{v}_i + \boldsymbol{v}_j)^t Q (\boldsymbol{v}_i - \boldsymbol{v}_j) = (\boldsymbol{e}_i + \boldsymbol{e}_j)^t C (\boldsymbol{e}_i - \boldsymbol{e}_j) = c_{ii} - c_{jj}. \end{aligned}$$

Moreover, for pairwise different indices $i, j, k, l$, the orthogonality yields

$$0 = (\pm \boldsymbol{e}_i \pm \boldsymbol{e}_j)^t (\pm \boldsymbol{e}_k \pm \boldsymbol{e}_l) = \pm c_{ik} \pm c_{il} \pm c_{jk} \pm c_{jl}.$$

Hence, the matrix $C$ is a multiple of the identity matrix and so $Q$ is a multiple of $Q_L$, which proves Theorem 5.38.

**5.5.3. Local covering optimality of the Leech lattice.** For the proof of Theorem 5.34 we use the fact that any non-empty set $\frac{1}{\sqrt{2n}} Q_\Lambda(n)$, $n > 0$, forms a spherical 11-design ([**69**, Ch. 7, Th. 23]) in the Euclidean space with inner product

$\langle\cdot,\cdot\rangle$ (defined by $Q_\Lambda$). Generally, a *spherical $t$-design* $X$ is a non-empty finite subset of the unit sphere $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$ satisfying

$$\frac{1}{\operatorname{vol} S^{d-1}} \int_{S^{d-1}} f(\boldsymbol{x}) d\boldsymbol{x} = \frac{1}{|X|} \sum_{\boldsymbol{x}\in X} f(\boldsymbol{x})$$

for every polynomial $f : \mathbb{R}^d \to \mathbb{R}$ of degree at most $t$. Here, $\operatorname{vol} S^{d-1}$ denotes the surface volume of $S^{d-1}$, not the volume of the enclosed ball. Equivalently, $X$ is a spherical $t$-design if and only if it satisfies the equalities (see [**260**, Th. 3.2]):

$$\begin{aligned} &\textstyle\sum_{\boldsymbol{x}\in X} \langle \boldsymbol{x}, \boldsymbol{y}\rangle^k = 0, && \text{for all } \boldsymbol{y} \in \mathbb{R}^d \text{ and all odd } k \le t,\\ &\textstyle\sum_{\boldsymbol{x}\in X} \langle \boldsymbol{x}, \boldsymbol{y}\rangle^k = \frac{1\cdot 3\cdots(k-1)}{d(d+2)\cdots(d+k-2)} |X| \langle \boldsymbol{y}, \boldsymbol{y}\rangle^{k/2}, && \text{for all } \boldsymbol{y} \in \mathbb{R}^d \text{ and all even } k \le t. \end{aligned}$$

For the proof of Theorem 5.34 the following spherical 2-design property is even sufficient:

LEMMA 5.39. *Let $Q \in \mathcal{S}^d_{>0}$ and let $X \subset \mathbb{R}^d$ denote a spherical 2-design with respect to the inner product given by $Q$. Then*

$$\sum_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{x}^t = \frac{|X|}{d} Q^{-1}.$$

Note that the Lemma shows in particular that a PQF $Q$ whose minimal vectors $\operatorname{Min} Q$ form a spherical 2-design, is strongly eutactic (see Section 3.3.5). In fact, the opposite is true as well. The minimal vectors $\operatorname{Min} Q$ of a strongly eutactic PQF $Q$ form a spherical 2-design (see [**174**, Corollary 16.1.3]).

PROOF OF LEMMA 5.39. Since $X$ forms a spherical 2-design, we have

$$\sum_{\boldsymbol{x}\in X} \left(\boldsymbol{x}^t Q \boldsymbol{y}\right)^2 = \frac{|X|}{d} (\boldsymbol{y}^t Q \boldsymbol{y}).$$

On the other hand

$$\sum_{\boldsymbol{x}\in X} \left(\boldsymbol{x}^t Q \boldsymbol{y}\right)^2 = \sum_{\boldsymbol{x}\in X} \boldsymbol{y}^t Q (\boldsymbol{x}\boldsymbol{x}^t) Q \boldsymbol{y} = \boldsymbol{y}^t Q \left(\sum_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{x}^t\right) Q \boldsymbol{y}.$$

Thus because both identities are valid for all $\boldsymbol{y} \in \mathbb{R}^d$ we derive the equality stated in the lemma. ☐

Now we are in a position to prove Theorem 5.34.

PROOF OF THEOREM 5.34. Let $L$ be a Delone simplex of $Q_\Lambda$ of type $\mathsf{A}_{24}$. We apply Proposition 5.21 to the orbit of $L$ under the automorphism group $\mathrm{Co}_0 = \{T \in \mathsf{GL}_{24}(\mathbb{Z}) : T^t Q_\Lambda T = Q_\Lambda\}$ of $Q_\Lambda$.

For every PQF $Q$ for which the simplices $TL$, $T \in \mathrm{Co}_0$, are Delone simplices, we have $\Theta(Q) \ge \sqrt{24^{24} \det F} \cdot \operatorname{vol}(B^{24})$ with $F = \frac{1}{25^2 |\mathrm{Co}_0|} \sum_{T\in \mathrm{Co}_0} \sum_{\boldsymbol{e}} \boldsymbol{e}\boldsymbol{e}^t$, where $\boldsymbol{e}$ runs through all the edge vectors of $TL$. Since $L$ has 275 edges in $Q_\Lambda(2)$ and 25 edges in $Q_\Lambda(3)$ and because of the transitivity of $\mathrm{Co}_0$ on $Q_\Lambda(2)$ and $Q_\Lambda(3)$ ([**69**, Ch. 10, Th. 27]) we get

$$F = \frac{1}{25^2 |\mathrm{Co}_0|} \left( \frac{275 |\mathrm{Co}_0|}{|Q_\Lambda(2)|} \sum_{\boldsymbol{e}\in Q_\Lambda(2)} \boldsymbol{e}\boldsymbol{e}^t + \frac{25 |\mathrm{Co}_0|}{|Q_\Lambda(3)|} \sum_{\boldsymbol{e}\in Q_\Lambda(3)} \boldsymbol{e}\boldsymbol{e}^t \right).$$

By Lemma 5.39 (applied to $Q_\Lambda/4$ and $Q_\Lambda/6$) this yields

$$F = \frac{1}{25^2}\left(\frac{275}{|Q_\Lambda(2)|}\cdot\frac{|Q_\Lambda(2)|}{6}Q_\Lambda^{-1} + \frac{25}{|Q_\Lambda(3)|}\cdot\frac{|Q_\Lambda(3)|}{4}Q_\Lambda^{-1}\right) = \frac{1}{2^2\cdot 3}Q_\Lambda^{-1}. \tag{5.14}$$

Since $\det Q_\Lambda^{-1} = 1$, it follows $\det F = \frac{1}{2^{48}3^{24}}$ and finally, by Proposition 5.21, we derive $\Theta(Q) \geq 4096\cdot \mathrm{vol}(B^{24}) = \Theta(Q_\Lambda)$. □

**5.5.4. Local packing-covering optimality.** Together with the local lattice packing optimality of the Leech lattice we immediately derive the local optimality of the Leech lattice with respect to the lattice packing-covering problem. However, using Proposition 5.23 we can also give a short direct proof.

Proof of Corollary 5.35. We apply Proposition 5.23 to the same orbit $TL$ ($T \in \mathrm{Co}_0$) of Delone simplices as in the proof of Theorem 5.34.

We use $F$ as in (5.14) and since $Q_\Lambda$ is eutactic we may use $Q_F = Q_\Lambda$ in Proposition 5.23. With $\lambda(Q_\Lambda) = 4$ we derive

$$\gamma(Q) \geq 2\sqrt{\frac{\frac{1}{2^2\cdot 3}\cdot 24}{4}} = \sqrt{2} = \gamma(Q_\Lambda)$$

for all PQFs $Q$ with Delone simplices $TL$, $T\in\mathrm{Co}_0$, which proves the assertion. □

For $\mathsf{E}_8$ the situation seems to differ from the covering case: Given a fixed Delone triangulation, the problem of finding the minimum $\gamma$ among all lattices with the same Delone triangulation can also be formulated as a convex optimization problem. In contrast to the covering case, the application of `MAXDET` to the triangulation constructed in Section 5.5.5 indicates that $\gamma(\mathsf{E}_8) = \sqrt{2}$ may in fact be a local optimum, as conjectured by Zong [**272**]. Note that he even conjectured that $\mathsf{E}_8$ gives the global optimum.

Note the remarkable fact that $d = 1, 2$ are the only known cases where the minimum covering density and the maximum packing density — and therefore the minimum packing-covering constant $\gamma_d = \min_L \gamma(L)$ — are attained by the same lattice. Maybe this is also yet another exceptional property of the beautiful Leech lattice?

**5.5.5. Local covering non-optimality of the $\mathsf{E}_8$ root lattice.** If we apply the method of Theorem 5.34 to $\mathsf{E}_8$ we get a local lower bound of $\sqrt{(8/9)^8}\cdot\mathrm{vol}(B^8) \approx 0.6243\cdot\mathrm{vol}(B^8)$. But $\Theta(\mathsf{E}_8) = \mathrm{vol}(B^8)$, since the circumradius of the regular cross polytopes in $\mathsf{E}_8$'s Delone subdivision is 1. Despite this gap, $\mathsf{E}_8$ could be a locally optimal covering lattice. The following proof of Theorem 5.36 shows that this is not the case. By Proposition 5.7 it suffices to find a PQF $Q$ with $\Theta(Q) < \Theta(Q_{\mathsf{E}_8})$ so that $\mathrm{Del}_{\mathbb{Z}^8}(Q)$ and $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ have a common refinement.

Below, we describe a systematic way to attain a refining Delone triangulation of $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$. Given such a triangulation we can find an approximation of the unique PQF minimizing $\Theta$ among all PQFs in the closure of its secondary cone (as described in Section 5.1). We give such an approximation at the end of this section and explain how to verify its properties with a simple computer program. Since we carry out the verification using exact arithmetic only, the proof of Theorem 5.36 is rigorous.

First, we describe all $\mathbb{Z}^8$-periodic triangulations refining the Delone subdivision of $Q_{\mathsf{E}_8}$, that is, all sets $\mathcal{P}$ of simplices satisfying the following conditions:

*no additional vertices*: all vertices of simplices $L \in \mathcal{P}$ lie in $\mathbb{Z}^8$.
*periodicity*: $\forall L \in \mathcal{P}, \boldsymbol{v} \in \mathbb{Z}^8 : \boldsymbol{v} + L \in \mathcal{P}$.
*face-to-face tiling*: $\forall L, L' \in \mathcal{P} : L \cap L' \in \mathcal{P}$.
*refinement*: $\forall L \in \mathcal{P}\ \exists L' \in \operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8}) : L \subseteq L'$.
*covering*: $\forall \boldsymbol{x} \in \mathbb{R}^8\ \exists L \in \mathcal{P} : \boldsymbol{x} \in L$.

Recall from Section 5.5.2 that $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ consists of simplices and cross polytopes only. Thus for a $\mathbb{Z}^8$-periodic triangulation refining $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ we have to specify how to split the 8-dimensional cross polytopes into simplices.

We say that two polytopes $P$ and $P'$ are $\mathbb{Z}^8$-*equivalent* if $P' = \boldsymbol{v} + P$ for some $\boldsymbol{v} \in \mathbb{Z}^8$. Every $\boldsymbol{w} \in Q_{\mathsf{E}_8}(4)$ defines a Delone cross polytope $P_{\boldsymbol{w}} = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{15}\}$ with $\boldsymbol{v}_0 = \boldsymbol{0}$, $\boldsymbol{v}_8 = \boldsymbol{w}$ and all other $\boldsymbol{v}_j \in Q_{\mathsf{E}_8}(2)$ with $\boldsymbol{v}_j + \boldsymbol{v}_{j+8} = \boldsymbol{v}_8$ (indices computed modulo 16).

Two $\boldsymbol{w}, \boldsymbol{w}' \in Q_{\mathsf{E}_8}(4)$ define $\mathbb{Z}^8$-equivalent cross polytopes if and only if $\boldsymbol{w}' \in \boldsymbol{w} + 2\mathbb{Z}^8$, because then the difference $\frac{1}{2}\boldsymbol{w}' - \frac{1}{2}\boldsymbol{w}$ of their centers is in $\mathbb{Z}^8$. Under this equivalence relation the set $Q_{\mathsf{E}_8}(4)$ splits into 135 classes, containing 8 pairs of mutually orthogonal vectors $\pm\boldsymbol{w}_1, \ldots, \pm\boldsymbol{w}_8$. Each of the $\boldsymbol{w}_i$ equivalent to $\boldsymbol{w}$ is a diagonal of $P_{\boldsymbol{w}}$, e.g. $\boldsymbol{w}_i = \boldsymbol{v}_i - \boldsymbol{v}_{i+8}$, $i = 0, \ldots, 7$. In the coordinate system introduced in (5.13) the 135 classes are (see [**69**, Ch. 6, §3]):

**:** 1 class: $\pm 2\boldsymbol{e}_1, \ldots, \pm 2\boldsymbol{e}_8$.
**:** 70 classes: 8 elements $\pm\boldsymbol{e}_a \pm \boldsymbol{e}_b \pm \boldsymbol{e}_c \pm \boldsymbol{e}_d$ with an even number of minus signs and 8 elements $\pm\boldsymbol{e}_e \pm \boldsymbol{e}_f \pm \boldsymbol{e}_g \pm \boldsymbol{e}_h$ with an even number of minus signs and with $\{a, \ldots, h\} = \{1, \ldots, 8\}$; or the same with an odd number of minus signs.
**:** 64 classes: 8 pairs of vectors of shape $(\pm\frac{3}{2}(\pm\frac{1}{2})^7)$ with odd number of minus signs, where the position of $\pm\frac{3}{2}$ is permuted to all 8 coordinates.

We can split each cross polytopes $P_{\boldsymbol{w}}$ into simplices in eight different ways by adding a diagonal. Without loss of generality we add the diagonal $\operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_8\}$ and split the cross polytope $P_{\boldsymbol{w}}$ into the 128 simplices $\operatorname{conv}\{\boldsymbol{0}, \boldsymbol{v}_8, \boldsymbol{v}_{j_1}, \ldots, \boldsymbol{v}_{j_7}\}$, where $j_k \in \{k, k+8\}$. Thus altogether we get $8^{135}$ different $\mathbb{Z}^8$-periodic triangulations refining $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$.

Now, which of these periodic triangulations are Delone triangulation for some PQF? To decide this, we take a closer look at the tiling $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ and at secondary cones $\boldsymbol{\Delta}(\mathcal{T})$ of Delone triangulations $\mathcal{T}$ refining $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$.

We already described the cross polytopes of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$. Centers of simplices of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ containing the origin are the 17280 vectors $\frac{1}{3}\boldsymbol{v}$, where $\boldsymbol{v}$ is a vector of $Q_{\mathsf{E}_8}(8)$ not in $2Q_{\mathsf{E}_8}(2)$. We say two polytopes are adjacent in the tiling, if they share a facet. Each simplex is adjacent to 9 cross polytopes and each cross polytope is adjacent to 128 simplices and 128 cross polytopes. A simplex and a cross polytope both containing the origin are adjacent if and only if the inner product, with respect to $Q_{\mathsf{E}_8}$, of their centers equals $\frac{5}{6}$. Two cross polytopes both containing the origin are adjacent if and only if the inner product of their centers equals $\frac{3}{4}$.

For some computations it is useful to have coordinates of vertices, with respect to the coordinate system (5.13): The vertices of the cross polytope $P$ defined by the center $\boldsymbol{e}_1$ are $\boldsymbol{0}, 2\boldsymbol{e}_1, \boldsymbol{e}_1 \pm \boldsymbol{e}_i$, $i = 2, \ldots, 8$. An adjacent Delone cross polytope is defined by the center $\boldsymbol{c} = (\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Its vertices are $\boldsymbol{0}, 2\boldsymbol{c}, \boldsymbol{e}_1 - \boldsymbol{e}_2, 2\boldsymbol{c} - (\boldsymbol{e}_1 - \boldsymbol{e}_2), \boldsymbol{e}_1 + \boldsymbol{e}_i, 2\boldsymbol{c} - (\boldsymbol{e}_1 + \boldsymbol{e}_i)$, $i = 3, \ldots, 8$. A Delone simplex adjacent to $P$ is defined by the center $\boldsymbol{c}' = (\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Its vertices are $\boldsymbol{0}, (\frac{1}{2}, \ldots, \frac{1}{2}), \boldsymbol{e}_1 + \boldsymbol{e}_i$, $i = 2, \ldots, 8$. Since the automorphism group of $\mathsf{E}_8$ acts transitively on vectors of squared length 4 and since the stabilizer of $\pm 2\boldsymbol{e}_1$ in $\mathsf{E}_8$'s automorphism group is the group generated by even sign changes and by permutations of the last 7 coordinates, the knowledge of the coordinates given is enough to describe the whole Delone subdivision.

By Voronoi's theory reviewed in Section 4.1.2, the secondary cones $\boldsymbol{\Delta}(\mathcal{T})$ are open polyhedral cones defined by linear forms $\langle N_{L,L'}, \cdot\rangle$ (regulators) on $\mathcal{S}^8$, one for each pair of adjacent simplices $L = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_8\}$, $L' = \operatorname{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_9\}$ in $\mathcal{T}$:

$$\boldsymbol{\Delta}(\mathcal{T}) = \left\{Q \in \mathcal{S}^8 : \langle N_{L,L'}, Q'\rangle > 0 \text{ for adjacent simplices } L, L' \in \mathcal{T}\right\}.$$

Recall that these are only finitely many linear conditions because of $N_{L+v,L'+v} = N_{L,L'}$ for all $\boldsymbol{v} \in \mathbb{Z}^8$.

For a triangulation $\mathcal{T}$ which is a refinement of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ we distinguish between three types of pairs of adjacent simplices $(L, L')$. In the first case one of the simplices is a simplex of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ and the other one is not. In the two other cases both simplices are not simplices of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$. In the second case they refine adjacent cross polytopes, in the third case they refine the same cross polytope. In the first two cases we have $\langle N_{L,L'}, Q_{\mathsf{E}_8}\rangle > 0$ and in the last case $\langle N_{L,L'}, Q_{\mathsf{E}_8}\rangle = 0$. Since $\mathsf{E}_8$ is rigid, the closures of secondary cones of Delone triangulations refining $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ cover a sufficient small neighborhood of the ray containing multiples of $Q_{\mathsf{E}_8}$. Thus, $\mathcal{T}$ is a Delone triangulation for some PQF refining $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ iff

$$\left\{Q \in \mathcal{S}^8 \;:\; \langle N_{L,L'}, Q\rangle > 0,\ (L, L') \text{ refining same cross polytope of } \operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})\right\}$$

is not empty.

The three types of regulators are easily computed, e.g. with help of the coordinates given above. In the first case, let $P = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{15}\}$ be a cross polytope of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ with the notational convention: $\boldsymbol{v}_8 \in Q_{\mathsf{E}_8}(4)$, $\boldsymbol{v}_i + \boldsymbol{v}_{i+8} = \boldsymbol{v}_8$. Let $L' = \operatorname{conv}\{\boldsymbol{v}'_0, \ldots, \boldsymbol{v}'_8\}$ be a simplex of $\operatorname{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ with $\boldsymbol{v}'_i = \boldsymbol{v}_i$, $i = 0, \ldots, 7$. Let $\boldsymbol{c}$ be the centroid of $P$ and $\boldsymbol{c}'$ be the centroid of $L'$. Then $\boldsymbol{c}' = \frac{1}{9}\left(\boldsymbol{v}'_0 + \ldots + \boldsymbol{v}'_8\right)$ and $\frac{1}{4}\boldsymbol{c} + \frac{3}{4}\boldsymbol{c}' = \frac{1}{8}(\boldsymbol{v}_0 + \cdots + \boldsymbol{v}_7)$. Suppose the edge $\operatorname{conv}\{\boldsymbol{v}_k, \boldsymbol{v}_{k+8}\}$, $k \in \{0, \ldots, 7\}$, belongs to $\mathcal{T}$. We have $\boldsymbol{c} = \frac{1}{2}(\boldsymbol{v}_k + \boldsymbol{v}_{k+8})$ and we derive $\frac{1}{8}(\boldsymbol{v}_k + \boldsymbol{v}_{k+8}) + \frac{1}{12}(\boldsymbol{v}'_8 + \boldsymbol{v}_0 + \cdots + \boldsymbol{v}_7) = \frac{1}{8}(\boldsymbol{v}_0 + \cdots + \boldsymbol{v}_7)$. Therefore we get the regulator

$$\langle N_{L,L'}, Q\rangle = Q[\boldsymbol{v}_k] + Q[\boldsymbol{v}_{k+8}] + \frac{2}{3}Q[\boldsymbol{v}'_8] - \frac{1}{3}Q[\boldsymbol{v}_0] - \cdots - \frac{1}{3}Q[\boldsymbol{v}_7]. \tag{5.15}$$

In the second case, let $P_1 = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{15}\}$ and $P_2 = \operatorname{conv}\{\boldsymbol{v}'_0, \ldots, \boldsymbol{v}'_{15}\}$ be two adjacent cross polytopes with the usual notational convention and with $\boldsymbol{v}_i = \boldsymbol{v}'_i$ for $i = 0, \ldots, 7$. Then the centers $c = \frac{1}{2}\boldsymbol{v}_8$ and $c' = \frac{1}{2}\boldsymbol{v}'_8$ of the cross polytopes satisfy the relation $\frac{1}{2}\left(c + c'\right) = \frac{1}{8}\left(\boldsymbol{v}_0 + \cdots + \boldsymbol{v}_7\right)$. Let us assume that the diagonals $\operatorname{conv}\{\boldsymbol{v}_k, \boldsymbol{v}_{k+8}\}$ and $\operatorname{conv}\{\boldsymbol{v}'_{k'}, \boldsymbol{v}'_{k'+8}\}$ with $k, k' \in \{0, \ldots, 7\}$ belong to $\mathcal{T}$. Then, since $c$, $c'$ are the centers of these diagonals, we derive $\boldsymbol{v}_k + \boldsymbol{v}_{k+8} + \boldsymbol{v}'_k + \boldsymbol{v}'_{k'+8} = \frac{1}{2}\boldsymbol{v}_0 + \cdots + \frac{1}{2}\boldsymbol{v}_7$. Therefore we get the regulator

$$\langle N_{L,L'}, Q\rangle = Q[\boldsymbol{v}_k] + Q[\boldsymbol{v}_{k+8}] + Q[\boldsymbol{v}'_{k'}] + Q[\boldsymbol{v}'_{k'+8}] - \frac{1}{2}Q[\boldsymbol{v}_0] - \cdots - \frac{1}{2}Q[\boldsymbol{v}_7]. \tag{5.16}$$

In the third case, let $P = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{15}\}$ be a cross polytope with the usual notational convention. Then adjacent simplices are of the form

$$L = \operatorname{conv}\{\boldsymbol{0}, \boldsymbol{v}_8, \boldsymbol{v}_{j_1}, \ldots, \boldsymbol{v}_{j_7}\}, \quad L' = \operatorname{conv}\{\boldsymbol{0}, \boldsymbol{v}_8, \boldsymbol{v}_{j'_1}, \ldots, \boldsymbol{v}_{j'_7}\},$$

| # | representative | orbit size | # | representative | orbit size |
|---|---|---|---|---|---|
| 1. | 2000000 | 1 | 11. | $\frac{1}{2}(3\overline{111}1111)$ | $\binom{7}{3}$ |
| 2. | 0200000 | 7 | 12. | $\frac{1}{2}(3\overline{11111}11)$ | $\binom{7}{5}$ |
| 3. | 11110000 | $\binom{7}{3}$ | 13. | $\frac{1}{2}(3\overline{1111111})$ | $\binom{7}{7}$ |
| 4. | $1\overline{1}110000$ | $7\binom{6}{2}$ | 14. | $\frac{1}{2}(13\overline{1}11111)$ | $7\binom{6}{1}$ |
| 5. | $1\overline{11}10000$ | $7\binom{6}{2}$ | 15. | $\frac{1}{2}(13\overline{111}111)$ | $7\binom{6}{3}$ |
| 6. | $1\overline{111}0000$ | $\binom{7}{3}$ | 16. | $\frac{1}{2}(13\overline{11111}1)$ | $7\binom{6}{5}$ |
| 7. | 01111000 | $\binom{7}{4}$ | 17. | $\frac{1}{2}(1\overline{3}111111)$ | $7\binom{6}{0}$ |
| 8. | $0\overline{1}111000$ | $7\binom{6}{3}$ | 18. | $\frac{1}{2}(1\overline{311}11111)$ | $7\binom{6}{2}$ |
| 9. | $0\overline{11}11000$ | $\frac{1}{2}\binom{7}{2}\binom{5}{2}$ | 19. | $\frac{1}{2}(1\overline{31111}111)$ | $7\binom{6}{4}$ |
| 10. | $\frac{1}{2}(3\overline{1}111111)$ | $\binom{7}{1}$ | 20. | $\frac{1}{2}(1\overline{3111111}1)$ | $7\binom{6}{6}$ |

TABLE 2. Orbits of length 2 vectors in $\mathsf{E}_8$. Minus signs are given by bars.

where $j_k, j'_k \in \{k, k+8\}$ and $j'_k = j_k + 8$ only for one $k \in \{1, \dots, 7\}$. Because of $\boldsymbol{v}_{j_k} + \boldsymbol{v}_{j_k+8} = \boldsymbol{v}_0 + \boldsymbol{v}_8$ we get seven regulators

$$\langle N_{L,L'}, Q\rangle = Q[\boldsymbol{v}_{j_k}] + Q[\boldsymbol{v}_{j_k+8}] - Q[\boldsymbol{v}_0] - Q[\boldsymbol{v}_8], \quad k = 1, \dots, 7. \tag{5.17}$$

Note that these conditions are equivalent to $Q[\boldsymbol{v}_8] < Q[\boldsymbol{v}_{j_k} - \boldsymbol{v}_{j_{k+8}}]$, $k = 1, \dots, 7$, which means that the chosen diagonal $\boldsymbol{v}_8$ is shorter than the other seven with respect to the metric induced by $Q$.

We tried to generate all Delone triangulations refining $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ by an exhaustive computer search. But this seems to be hopeless since they are far to many. So we decided to generate a Delone triangulation which has a fairly large symmetry group. For this we choose a subgroup $G$ of $Q_{\mathsf{E}_8}$'s automorphism group which, in the coordinate system (5.13), is generated by permutations of the last 7 coordinates and by the involution $\boldsymbol{x} \mapsto -\boldsymbol{x}$.

PROPOSITION 5.40. *There are exactly four $\mathbb{Z}^8$-periodic triangulations refining $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ invariant under the group $G$. Exactly two of them are Delone triangulations and both are equivalent under the action of $Q_{\mathsf{E}_8}$'s full automorphism group.*

PROOF. To show that there is essentially one Delone triangulation with the prescribed symmetries, we will again work with the coordinate system (5.13). In Table 2 we list the orbits of squared length 4 vectors under the action of $G$.
To define a $\mathbb{Z}^8$-periodic triangulation refining $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$ we have to choose a collection of orbits $(O_i)_{i\in I}$, $I \subseteq \{1, \dots, 20\}$, so that for every of the 135 classes of possible diagonals $C$ we have $\left|\bigcup_{i\in I} O_i \cap C\right| = 2$. This restriction immediately gives $\{2, 4, 5, 8, 9, 14, 15, 16, 17, 18, 19\} \cap I = \emptyset$. For example, the four vectors $\pm 02000000$, $\pm 00200000$ are in $O_2 \cap C$. On the other hand we have to have $\{1, 6, 11, 12, 13\} \subseteq I$. Now there are two binary choices left: either we have $3 \in I$ or $7 \in I$ and either we have $10 \in I$ or $20 \in I$. From these four triangulations only those given by $I_1 = \{1, 3, 6, 10, 11, 12, 13\}$ and $I_2 = \{1, 6, 7, 11, 12, 13, 20\}$ are Delone triangulations: Under the prescribed symmetry we can assume that there are numbers $\alpha$, $\beta$, $\gamma$, $\delta$ with

$$\alpha = (\boldsymbol{e}_1, \boldsymbol{e}_1),\ \ \beta = (\boldsymbol{e}_1, \boldsymbol{e}_i),\ \ \gamma = (\boldsymbol{e}_i, \boldsymbol{e}_i),\ \ \delta = (\boldsymbol{e}_i, \boldsymbol{e}_j),\ \ i = 2, \dots, 8, j = i+1, \dots, 8.$$

Suppose we choose orbit 3. By (5.17) this implies the inequality

$$(\frac{1}{2}(11111111), \frac{1}{2}(1111\overline{1111})) = \frac{1}{4}(\alpha + 6\beta - \gamma - 6\delta) < 0,$$

then choosing orbit 20 implies

$$(\tfrac{1}{2}(1\overline{1}000000), \tfrac{1}{2}(\overline{11111111})) = \tfrac{1}{4}(-\alpha - 6\beta + \gamma + 6\delta) < 0,$$

yielding a contradiction. Hence we have to choose orbit 10 instead. A similar calculation shows that if we choose orbit 7, then we have to choose orbit 20. To see that these triangulations are Delone triangulations we still have to give a PQF satisfying all regulators in (5.17). We postpone this to the end of this section.

By applying the transformation $HAH$, where $A$ exchanges the first and fifth coordinate and their signs, and $H$ is the transformation $H = \mathrm{diag}(H_4, H_4)$ we see that both Delone triangulations are equivalent. The transformation $HAH$ is an element of $\mathsf{E}_8$'s automorphism group exchanging the relevant orbits of vectors of size 2 by $O_1 \leftrightarrow O_{13}$, $O_3 \leftrightarrow O_7$, $O_6 \leftrightarrow O_{11}$, $O_{10} \leftrightarrow O_{20}$, $O_{12} \leftrightarrow O_{12}$. □

Given the Delone triangulation $\mathcal{T}$ refining $\mathrm{Del}_{\mathbb{Z}^8}(Q_{\mathsf{E}_8})$, attained in this way or another, we can compute an approximation of the unique PQF minimizing the covering density $\Theta$ among all PQFs in the closure of its secondary cone, using our software `coop` (see Section 5.1.5 and our package [**285**]). We found a PQF $\tilde{Q} = 34229189769Q_1 - 17121746137Q_2$ with covering density $\Theta \approx 3.2012$. Here

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4/7 & -2/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 & 4/3 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad \text{and}$$

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 7/2 \\ 1 & 0 & -2/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 & 4/3 & 0 \\ 7/2 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

form a basis of the subspace of $\mathcal{S}^8$ invariant under the group $G$ intersected with the subspace given by regulators $\langle N_{L,L'}, \cdot\rangle$ of (5.15), (5.16), (5.17) with $\langle N_{L,L'}, \tilde{Q}\rangle = 0$. To give a rigorous proof of Theorem 5.36 we have to verify that one of the two triangulations we constructed in Section 5.5.5 is a refining Delone triangulation of $\tilde{Q}$'s Delone subdivision and that its covering density is at most 3.2013. To do this we supply the `MAGMA` program `newcovering8.m` available from the `arXiv.org` e-print archive. To access it, download the source files for the paper `math.MG/0405441`.

The program `newcovering8.m` first verifies that there is a PQF, called $Q_{\texttt{interior}}$, which strictly satisfies the inequalities (5.17) for the triangulation given by $I_1$ (see proof of Proposition 5.40). By this we know that the triangulation is a Delone triangulation. Then the program checks that $\tilde{Q}$ satisfies all inequalities (5.15), (5.16), (5.17) given by regulators. This verifies that the Delone triangulation is in fact a refinement of $\tilde{Q}$'s Delone subdivision. Finally we compute the circumradii of a representative system of simplices with respect to the inner product $(\cdot,\cdot)$ induced by $\tilde{Q}$. The squared circumradius $R^2$ of the simplex $L = \operatorname{conv}\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_8\}$ is given by (5.7). All these evaluations involve only rational arithmetic and they can be carried out on a usual personal computer in less than 15 minutes.

**5.5.6. A new eight-dimensional sphere covering.** The following came out of a computational analysis of $\tilde{Q}$ and $\boldsymbol{\Delta}(\mathcal{T})$: The PQF $\tilde{Q}$ lies on the boundary of the secondary cone $\boldsymbol{\Delta}(\mathcal{T})$. The closure of $\boldsymbol{\Delta}(\mathcal{T})$ has 428 facets and only one of these facets does not contain $Q_{\mathsf{E}_8}$. We applied the optimization to the Delone triangulation which belongs to the secondary cone adjacent to this facet. There we found a PQF with covering density $\Theta \approx 3.1423$, which is the best known covering density in dimension 8 so far. An approximation of its Gram matrix is

$$\begin{pmatrix}
3.1917 & -1.5938 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.8051 \\
-1.5938 & 1.9158 & -1.1726 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \\
-0.0000 & -1.1726 & 2.3453 & -1.1726 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\
0.0000 & -0.0000 & -1.1726 & 2.3453 & -1.1726 & -0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & 0.0000 & -1.1726 & 2.3453 & -1.1726 & -0.0000 & -0.0000 \\
-0.0000 & -0.0000 & -0.0000 & -0.0000 & -1.1726 & 2.3453 & -1.1726 & 0.0000 \\
-0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -1.1726 & 2.3453 & -0.0000 \\
0.8051 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & 1.6103
\end{pmatrix}.$$

It seems that the PQF we approximated by $\tilde{Q}$ is not locally optimal. This would follow by Proposition 5.7, if we knew that the by $\tilde{Q}$ approximated PQF lies on the boundary of $\boldsymbol{\Delta}(\mathcal{T})$. In any case we are left with the open problem to find a globally best covering lattice in dimension 8. Currently we do not know where to search for such a lattice.

## 5.6. Local lattice covering maxima and pessima

For the lattice sphere packing problem we have finitely many local density maxima, but no local density minima (on the space $\mathbb{R}_{>0}\backslash\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$ of similarity classes of $d$-dimensional lattices). We show in this section that the lattice covering problem is quite different. Although we have only finitely many local density minima, it turns out that there also exist local density maxima. Here, we say a lattice and a corresponding PQF $Q$ is a *local lattice (sphere) covering maximum* if any local change of $\mathbb{R}_{>0}Q$ in $\mathbb{R}_{>0}\backslash\mathcal{S}^d_{>0}$ decreases the lattice covering density $\Theta$. Moreover, it turns out that there exist lattices as the root lattice $\mathsf{E}_8$, which do not give a local lattice covering maximum, but for which almost all local changes yield a decrease of the density function. Following a suggestion of Peter McMullen, we call such lattices *covering pessima*.

This section is structured as follows: From a local analysis in Section 5.6.1, we obtain in Section 5.6.2 and Section 5.6.3 a necessary (Theorem 5.42) and a sufficient condition (Theorem 5.45) for a local lattice covering maximum. From these conditions we obtain a complete classification of local lattice sphere covering maxima up to dimension 6 (see Corollaries 5.44 and 5.46). A class of lattice covering pessima (including the root lattices $\mathsf{D}_4$ and $\mathsf{E}_8$) is obtained in Section 5.6.4.

**5.6.1. Local analysis of the covering density function.** In this section we start by further studying the behavior of the covering density function $\Theta$ in the neighborhood of a given PQF $Q$. In Lemma 5.41 we identify the regions where the covering density function increases and where it decreases.

Let us first collect the necessary background from previous sections. Recall from Section 5.2.1 and from (3.15) that $\mathbf{D}_\alpha = \{Q \in \mathcal{S}^d_{>0} : \det Q \geq \alpha\}$ is a strictly convex set and that the tangent hyperplane of $\mathbf{D}_{\det Q}$ at $Q$ is

$$\mathcal{H}_{\det}(Q) = \{Q' \in \mathcal{S}^d : \langle Q' - Q, Q^{-1}\rangle = 0\}.$$

Recall the definition of the smooth hypersurface

$$\mathcal{B}_{L,=\beta} = \{Q \in \mathcal{S}^d_{>0} : \mu_L(Q) = \beta\},$$

for a $d$-dimensional simplex $L$ with vertices in $\mathbb{Z}^d$ (one of them the origin) and a positive $\beta > 0$. ($\mu_L(Q)$ denotes the squared circumradius of $L$ with respect to $Q$.) The tangent hyperplane of $\mathcal{B}_{L,=\mu_L(Q)}$ at $Q$ is

$$\mathcal{H}_L(Q) = \{Q' \in \mathcal{S}^d : \langle Q' - Q, g_{L,\mu_L(Q)}(Q)\rangle = 0\},$$

where $g_{L,\beta}(Q) = (\operatorname{grad} |\operatorname{BR}_{L,\beta}|)(Q)$ (see Section 5.2.1).

The set $\mathcal{B}_{L,\leq\beta}$ of all PQFs with $\mu_L(Q) \leq \beta$ is convex and $-g_{L,\mu_L(Q)}(Q)$ is the outer normal vector of it at $Q$. The hyperplane $\mathcal{H}_L(Q)$ divides $\mathcal{S}^d$ into the closed half space

$$\mathcal{H}_L^{\geq 0}(Q) = \{Q' \in \mathcal{S}^d : \langle Q' - Q, -g_{L,\mu_L(Q)}(Q)\rangle \geq 0\}$$

and the open half space $\mathcal{H}_L^{<0}(Q) = \mathcal{S}^d \setminus \mathcal{H}_L^{\geq 0}(Q)$.

For a Delone polytope $P$ with vertices in $\mathbb{Z}^d$, we define the *$P$-covering density function* by

$$\Theta_P(Q) = \sqrt{\frac{\mu_P(Q)^d}{\det Q}} \cdot \operatorname{vol}(B^d).$$

Then in particular

$$\Theta(Q) = \max_{P \in \operatorname{Del}_{\mathbb{Z}^d}(Q)} \Theta_P(Q). \tag{5.18}$$

Note that all simplices $L$ with $\operatorname{vert} L \subseteq \operatorname{vert} P$ satisfy $\Theta_L(Q) = \Theta_P(Q)$.

The following lemma tells us where $\Theta_L$ increases and where it decreases.

Lemma 5.41. *Let $Q \in \mathcal{S}^d_{>0}$ be a PQF and let $L = \operatorname{conv}\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ be a $d$-dimensional simplex with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in \mathbb{Z}^d$. For a PQF $Q' \in \mathcal{H}_{\det}(Q)$ we distinguish between two cases:*

(a) *If $Q' \in \mathcal{H}_L^{\geq 0}(Q)$, then $\Theta_L(Q') > \Theta_L(Q)$ whenever $Q' \neq Q$.*
(b) *If $Q' \in \mathcal{H}_L^{<0}(Q)$, then there exists an $\varepsilon > 0$ so that*

$$\Theta_L((1-\varepsilon)Q + \varepsilon Q') < \Theta_L(Q).$$

Proof. We consider the ray in $\mathcal{S}^d$ starting at the origin and going through $Q'$. On this ray the $L$-covering density is equal to $\Theta_L(Q')$. We define the number $\alpha_D$ by the property that $\det(\alpha_D Q') = \det Q$. Correspondingly, we define the numbers $\alpha_{\mathcal{H}}$ and $\alpha_{\mathcal{B}}$ such that $\alpha_{\mathcal{H}} Q' \in \mathcal{H}_L(Q)$ and $\alpha_{\mathcal{B}} Q' \in \mathcal{B}_{L,=\mu(Q)}$.

What are the possible relations between these three numbers? If $Q = Q'$, then they are all equal to 1. Otherwise we always have $1 < \alpha_D$ and $\alpha_{\mathcal{B}} < \alpha_{\mathcal{H}}$ because $\mathbf{D}_{\det Q}$ and $\mathcal{B}_{L,\leq\mu(Q)}$ are convex and have a smooth boundary.

We distinguish between the two cases of the lemma:

(a) If $Q' \in \mathcal{H}_L^{\geq 0}(Q)$, then $\alpha_{\mathcal{H}} \leq 1$. In this case the PQF $\alpha_{\mathcal{B}} Q'$ satisfies $\mu_L(\alpha_{\mathcal{B}} Q') = \mu_L(Q)$ and $\det(\alpha_{\mathcal{B}} Q') < \det Q$ whenever $Q \neq Q'$ which settles (a).

(b) If $Q' \in \mathcal{H}_L^{<0}(Q)$, then $\alpha_{\mathcal{H}} > 1$. In this case the convex sets $\mathcal{D}_{\det Q}$ and $\mathrm{BR}_{L,\leq \mu_L(Q)}$ have a full-dimensional intersection because they have a smooth boundary. So there is an $\varepsilon > 0$, so that for $Q'$ replaced by $(1-\varepsilon)Q + \varepsilon Q'$ we have

$$1 < \alpha_D < \alpha_{\mathcal{B}} < \alpha_{\mathcal{H}}.$$

Then, $\det(\alpha_D Q') = \det Q$ and $\mu_L(\alpha_D Q') > \mu_L(Q)$, and hence $\Theta_L(Q') < \Theta_L(Q)$, which settles (b). $\square$

In the following two sections we use Lemma 5.41 to give computationally testable, necessary and sufficient criteria for a given PQF to attain a local maximum of the lattice covering density $\Theta$ on $\mathbb{R}_{>0} \backslash \mathcal{S}^d_{>0}$.

**5.6.2. Necessary conditions for a local lattice covering maximum.** Recall some properties of the secondary cone $\mathbf{\Delta}(\mathbb{Z}^d, P)$ of a $d$-dimensional polytope with vertices in $\mathbb{Z}^d$: It is a relative open polyhedral cone (see Proposition 4.22), containing all PQFs for which $P$ is a Delone polytope. In fact, it is a face of every Baranovskii cones $\mathbf{\Delta}(\mathbb{Z}^d, L)$, where $L$ is a $d$-dimensional simplex $L$ with $\operatorname{vert} L \subseteq \operatorname{vert} P$ (see Section 4.4.2). A Delone polytope is called extreme if $\mathbf{\Delta}(\mathbb{Z}^d, P)$ has dimension 1, that is, if its secondary cone is a ray.

THEOREM 5.42. *Suppose $Q \in \mathcal{S}^d_{>0}$ attains a local maximum of the lattice covering density $\Theta$. Then every Delone polytope $P \in \operatorname{Del}_{\mathbb{Z}^d}(Q)$ with $\Theta_P(Q) = \Theta(Q)$ is an extreme Delone polytope.*

PROOF. Suppose there exists a Delone polytope $P \in \operatorname{Del}_{\mathbb{Z}^d}(Q)$ with $\Theta_P(Q) = \Theta(Q)$ which is not an extreme Delone polytope. Then the relative open polyhedral set $\mathbf{\Delta}(\mathbb{Z}^d, P)$ containing $Q$ is not a ray. Factoring out positive scaling, we know that the function $\Theta_P$ is a strictly convex function on the compact set $\mathbb{R}_{>0} \backslash \overline{\mathbf{\Delta}(\mathbb{Z}^d, P)}$. In it, $\mathbb{R}_{\geq 0} Q$ is a relative interior point. Thus $Q$ cannot attain a local maximum of $\Theta_P$, and because of (5.18), it also cannot attain a local maximum of $\Theta$. $\square$

As immediate consequences we derive the following corollaries. First, extreme Delone polytopes can only occur for rigid PQFs, that is for PQFs with $\mathbf{\Delta}(\operatorname{Del}_{\mathbb{Z}^d}(Q))$ being a ray. Since for a given dimension $d$ there exist only finitely many $\mathsf{GL}_d(\mathbb{Z})$-inequivalent secondary cones, there exist only finitely many local lattice covering maxima for a fixed dimension $d$. In lattice terminology this gives:

COROLLARY 5.43. *A lattice giving a local maximum for the covering density function is rigid. In particular, up to similarities, there exist only finitely many local lattice covering maxima for a given dimension $d$.*

Since there exist no extreme Delone polytopes in dimensions $d = 2, \ldots, 5$ (see [**104**]; cf. Section 4.4.1), there exist no local maxima for $\Theta$ in these dimensions.

COROLLARY 5.44. *There exists no local lattice covering maxima in dimension $d = 2, \ldots, 5$.*

Note that the one-dimensional integer lattice $\mathbb{Z}$ trivially attains a local maximum (and minimum) of $\Theta$.

**5.6.3. Sufficient conditions for a local lattice covering maximum.** We want to give a sufficient and computable criterion for a local lattice covering maximum. Assume a PQF $Q$ satisfies the necessary condition of Theorem 5.42. We consider all $d$-dimensional simplices $L$ with $\operatorname{vert} L \subset \operatorname{vert} P$ for some extreme Delone polytope $P \in \operatorname{Del}_{\mathbb{Z}^d}(Q)$ with $\mu(Q) = \mu_P(Q)$. Then in particular $\mu(Q) = \mu_L(Q)$. Recall that the Baranovskii cone $\mathbf{\Delta}(\mathbb{Z}^d, L)$ is the cone of all PQFs that have $L$ as a Delone simplex. Assuming $\mathbf{\Delta}(\mathbb{Z}^d, L)$ is non-empty, we know that $Q$ lies on a ray of the closed polyhedral cone $\overline{\mathbf{\Delta}(\mathbb{Z}^d, L)}$. This ray is the secondary cone $\mathbf{\Delta}(\mathbb{Z}^d, L)$. Let $\mathcal{N}_{\mathbb{Z}^d,L}(Q)$ denote the outer normal cone of $\mathbf{\Delta}(\mathbb{Z}^d, L)$ at $Q$. (The normals generating $\mathcal{N}_{\mathbb{Z}^d,L}(Q)$ are of type $-N_{\operatorname{vert} L,\boldsymbol{w}}$ with $\boldsymbol{w} \in \operatorname{vert} P \setminus \operatorname{vert} L$; see (4.3) and Proposition 4.8.)

We want to study the change of $\Theta$, if we change $Q$ to some $Q'$ in $\overline{\mathbf{\Delta}(\mathbb{Z}^d, L)}$, close to $Q$. We may assume $Q' \in \mathcal{H}_{\det}(Q)$, which allows us to apply Lemma 5.41. In order to state a sufficient computable criterion, let $\pi_{\mathcal{H}}$ be the orthogonal projection of $\mathcal{S}^d$ onto the hyperplane

$$\mathcal{H} = \{Q' \in \mathcal{S}^d : \langle Q', Q^{-1}\rangle = 0\},$$

parallel to $\mathcal{H}_{\det}(Q)$.

Theorem 5.45. *Let $Q \in \mathcal{S}^d_{>0}$ be a PQF. Suppose that every Delone polytope $P$ in $\operatorname{Del}_{\mathbb{Z}^d}(Q)$ with $\mu_P(Q) = \mu(Q)$ is extreme. Assume further that for every of these extreme Delone polytopes with $\mathbf{0} \in \operatorname{vert} P$ and every d-dimensional simplex $L$, with $\mathbf{0} \in \operatorname{vert} L \subseteq \operatorname{vert} P$ and $\mathbf{\Delta}(\mathbb{Z}^d, L) \neq \emptyset$, we have*

$$\pi_{\mathcal{H}}(-g_{L,\mu(Q)}(Q)) \in \operatorname{relint} \pi_{\mathcal{H}}(\mathcal{N}_{\mathbb{Z}^d,L}(Q)). \tag{5.19}$$

*Then $Q$ attains a local maximum of the lattice covering density function.*

Proof. Let $Q' \notin \mathbb{R}_{\geq 0} Q$ be a PQF in a sufficiently small neighborhood of $Q$. We assume in particular that $\operatorname{Del}_{\mathbb{Z}^d}(Q')$ is a refinement of $\operatorname{Del}_{\mathbb{Z}^d}(Q)$ and that $P' \in \operatorname{Del}_{\mathbb{Z}^d}(Q')$ satisfies $\Theta_{P'}(Q') = \Theta(Q')$. Then (by $Q'$ sufficiently close to $Q$) there exists at least one simplex $L$ such that $Q, Q' \in \overline{\mathbf{\Delta}(\mathbb{Z}^d, L)}$ and such that $\Theta_L(Q) = \Theta(Q)$ and $\operatorname{vert} L \subseteq \operatorname{vert} P' \subseteq \operatorname{vert} P$ for some extreme Delone polytope $P \in \operatorname{Del}_{\mathbb{Z}^d}(Q)$.

We are going to apply Lemma 5.41. We scale $Q'$ so that it lies on the hyperplane $\mathcal{H}_{\det}(Q) = \mathcal{H} + Q$. Then, $Q' - Q \in \mathcal{H}$. Let $\mathcal{N}_{\mathbb{Z}^d,L}(Q) = \operatorname{cone}\{N_i : i = 1, \dots n\}$ be generated by outer normals $N_1, \dots, N_n$. By condition (5.19) we know that

$$\pi_{\mathcal{H}}(-g_{L,\mu_L(Q)}(Q)) = \sum_{i=1}^{n} \alpha_i N_i$$

with suitable $\alpha_i > 0$. Since $\langle Q', N_i\rangle \leq 0$ for $i = 1, \dots, n$, with strict inequality for at least one index $i$, we obtain

$$\langle Q' - Q, \pi_{\mathcal{H}}(-g_{L,\mu_L(Q)}(Q))\rangle < 0.$$

Because of $\langle Q' - Q, Q^{-1}\rangle = 0$ this immediately gives

$$\langle Q' - Q, -g_{L,\mu_L(Q)}(Q)\rangle < 0.$$

So we are in case (b) of Lemma 5.41 saying that when $Q'$ is sufficiently close to $Q$, then

$$\Theta(Q') = \Theta_L(Q') < \Theta_L(Q) = \Theta(Q).$$

□

By a result of Dutour [**86**], the Schläfli polytope is the only extreme Delone polytope in dimension 6. It determines the root lattice $\mathsf{E}_6$, which thereby is the only candidate for a local lattice covering maximum. In fact, up to isometries the Schläfli polytope is the only Delone polytope of $\mathsf{E}_6$. So the first requirement of Theorem 5.45 is satisfied. Thus in order to show that $\mathsf{E}_6$ is a local lattice covering maximum it remains to check condition (5.19). The Schläfli polytope has 27 vertices and it is known that all simplices refining it are Delone simplices with a non-empty Baranovskii cone. Using `MAGMA` (see `http://fma2.math.uni-magdeburg.de/~latgeo/covering_max_e6.m`) we find that there are precisely 101360 full dimensional Delone simplices with this property. All of them turn out to satisfy the criterion of Theorem 5.45. Thus we obtain the following corollary.

Corollary 5.46. *The root lattice* $\mathsf{E}_6$ *gives the only local lattice covering maximum in dimension* 6.

Going to higher dimensions, it is more and more difficult to apply Theorem 5.45, due to the increasing number of Delone simplices which one has to consider. However, it is sufficient to apply the criterion of Theorem 5.45 to only one simplex of each orbit under the symmetry group of the given lattice. For example, according to Dutour Sikirić (private communication), under the symmetries of $\mathsf{E}_6$, there exist only 31 orbits of 6-dimensional Delone simplices refining the Schläfli polytope.

In the forthcoming paper [**94**] we give another sufficient criterion, which is easier to apply in higher dimensions: $Q \in \mathcal{S}^d_{>0}$ gives a local lattice covering maximum, if every Delone polytope $P$ attaining $\mu(Q)$ is extreme, and if for all of them there exist $\alpha_{\boldsymbol{v}} > 0$ for $\boldsymbol{v} \in \operatorname{vert} P$ such that

(1) $1 = \sum_{\boldsymbol{v}\in \operatorname{vert} P} \alpha_{\boldsymbol{v}}$,

(2) $0 = \sum_{\boldsymbol{v}\in \operatorname{vert} P} \alpha_{\boldsymbol{v}}(\boldsymbol{v}-\boldsymbol{c})$,

(3) $\dfrac{\mu_P(Q)}{d} Q^{-1} = \sum_{\boldsymbol{v}\in \operatorname{vert} P} \alpha_{\boldsymbol{v}}(\boldsymbol{v}-\boldsymbol{c})(\boldsymbol{v}-\boldsymbol{c})^t$.

Here, $\boldsymbol{c}$ denotes the center of $P$'s circumsphere with respect to $Q$. The latter conditions are in particular satisfied if the vertices of $P$ form a spherical 2-design with respect to $Q$.

This new criterion can be applied not only to $\mathsf{E}_6$, but also to many other lattices. By it we show in [**94**] that the two known extreme Delone polytopes in dimension 7 (the Gosset polytope defining the root lattice $\mathsf{E}_7$ and a 35-tope constructed by Erdahl and Rybnikov [**106**]) determine local lattice covering maxima. Furthermore, it turns out that the Barnes-Wall lattice is a local lattice covering maximum and we even find an infinite series of lattice covering maxima. We think this topic deserves further attention.

**5.6.4. Covering Pessima.** Aside of the existence of local lattice covering maxima, we find another, closely connected phenomenon of the lattice covering density function: The existence of lattices and PQFs for which almost every local change decreases the covering density. We call a PQF $Q$ (and a corresponding lattice) a *covering pessimum*, if the set of directions $N \in \mathcal{S}^d$ with $\Theta(Q + \epsilon N)$ increasing for small enough $\epsilon$ is a non-empty *measure zero set* (with respect to the usual Lebesgue measure on $\mathcal{S}^d$).

Such covering pessima occur whenever all Delone polytopes attaining the covering radius satisfy condition (5.19) in Theorem 5.45, and if in addition none of them is a simplex, and at least one is not extreme.

THEOREM 5.47. *Let $Q \in \mathcal{S}^d_{>0}$ be a PQF. Suppose that Delone polytopes $P$ in $\mathrm{Del}_{\mathbb{Z}^d}(Q)$ with $\mu_P(Q) = \mu(Q)$ are not simplices, and that at least one of them is not extreme. Assume further that for every of these Delone polytopes with $\mathbf{0} \in \mathrm{vert}\, P$ and every $d$-dimensional simplex $L$, with $\mathbf{0} \in \mathrm{vert}\, L \subseteq \mathrm{vert}\, P$ and $\mathbf{\Delta}(\mathbb{Z}^d, L) \neq \emptyset$, we have*

$$\pi_{\mathcal{H}}(-g_{L,\mu(Q)}(Q)) \in \mathrm{relint}\, \pi_{\mathcal{H}}(\mathcal{N}_{\mathbb{Z}^d,L}(Q)). \tag{5.20}$$

*Then $Q$ is a covering pessimum.*

PROOF. The argument in the proof of Theorem 5.42 shows that the covering density increases within the secondary cones $\mathbf{\Delta}(\mathbb{Z}^d, P)$, where $P$ is a non-extreme Delone polytope with $\mu_P(Q) = \mu(Q)$. Since we assume that such $P$ are not simplices, $\mathbf{\Delta}(\mathbb{Z}^d, P)$ and hence the corresponding directions of change (preserving $P$ as a Delone polytope) are measure zero sets.

In directions outside these measure zero sets, the arguments in the proof of Theorem 5.45 show (by (5.20)) that the lattice covering density decreases in direction $N = Q' - Q$, unless $\langle Q', N_i \rangle = 0$, for $i = 1, \ldots, n$ (where $N_1, \ldots, N_n$ are the outer normals of $\overline{\mathbf{\Delta}(\mathbb{Z}^d, L)}$ at $Q$). Here, $L$ is a Delone simplex with $\mathrm{vert}\, L \subseteq \mathrm{vert}\, P$ for some Delone polytope $P$ in $\mathrm{Del}_{\mathbb{Z}^d}(Q)$. If $P$ is extreme, this cannot happen for $Q' \notin \mathbb{R}_{\geq 0} Q$; if $P$ is not extreme, it could. However, by our assumption that $P$ is not a simplex, it can only occur on a measure zero set, namely on $\{X \in \mathcal{S}^d : \langle X, N_i \rangle = 0, i = 1, \ldots n\}$. $\square$

As a corollary we find that the root lattices $\mathsf{D}_4$ and $\mathsf{E}_8$ give covering pessima. Both lattices have the property that their covering radius is attained only by *regular cross polytopes*, that is a polytope similar to $\mathrm{conv}\{\pm \boldsymbol{e}_i : i = 1, \ldots, d\}$. Note that this applies also to the lattice $\mathbb{Z}^2$.

COROLLARY 5.48. *Let $d \geq 2$ and $Q \in \mathcal{S}^d_{>0}$. If all Delone polytopes $P$ of $Q$ with $\mu_P(Q) = \mu(Q)$ are regular cross polytopes (with respect to $Q$), then $Q$ is a covering pessimum.*

PROOF. We change coordinates such that $P = \boldsymbol{e}_1 + \mathrm{conv}\{\pm \boldsymbol{e}_i : i = 1, \ldots, d\}$ is a Delone polytope with respect to $\mathrm{id}_d$ and a lattice $\Lambda$ ($\Lambda = A\mathbb{Z}^d$ for suitable $A \in \mathsf{GL}_d(\mathbb{R})$ with $Q = A^t A$). Assume $L$ is a $d$-dimensional simplex refining $P$. Without loss of generality we assume that $\mathrm{conv}\{\mathbf{0}, 2\boldsymbol{e}_1\}$ is an edge of $L$. (The other $d-1$ vertices of $L$ are then one out of each pair $\pm \boldsymbol{e}_i$, $i = 2, \ldots, d$.)

The outer normal cone $\mathcal{N}_{\Lambda,L}$ of $\mathbf{\Delta}(\Lambda, L)$ at id is generated by $N_i = \boldsymbol{e}_1 \boldsymbol{e}_1^t - \boldsymbol{e}_i \boldsymbol{e}_i^t$, $i = 2, \ldots, d$. Hence for $Q' \in \mathbf{\Delta}(\Lambda, L)$ we have $\langle N_i, Q' \rangle = Q'[\boldsymbol{e}_1] - Q'[\boldsymbol{e}_i] < 0$. We may interpret this saying "if $L$ is a Delone simplex, then the diagonal $\mathrm{conv}\{\mathbf{0}, 2\boldsymbol{e}_1\}$ of $P$ is shorter than the diagonal $\boldsymbol{e}_1 + \mathrm{conv}\{\pm \boldsymbol{e}_i\}$".

Because of $\mathrm{id}_d^{-1} = \mathrm{id}_d$ we find $\mathcal{H} = \{X \in \mathcal{S}^d : \operatorname{trace} X = 0\}$. To compute $g_{L,1}(\mathrm{id}_d) = (\operatorname{grad} |\operatorname{BR}_{L,1}|)(\mathrm{id}_d)$ we first observe by elementary operations

$$|\mathrm{BR}_{L,1}(Q)| = 4 \begin{vmatrix} 4-4q_{11} & 0 & q_{22}-q_{11} & \cdots & q_{dd}-q_{11} \\ 0 & q_{11} & q_{12} & \cdots & q_{1d} \\ q_{22}-q_{11} & q_{12} & q_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ q_{dd}-q_{11} & q_{1d} & \cdots & \cdots & q_{dd} \end{vmatrix}.$$

Applying the chain rule to $f \circ g = |\mathrm{BR}_{L,1}|$ with $f = 4 \det$ and $g : \mathcal{S}^d \to \mathbb{R}^{(d+1)\times(d+1)}$ assigning the matrix above yields

$$g_{L,1}(\mathrm{id}_d) = -16 \boldsymbol{e}_1 \boldsymbol{e}_1^t.$$

We may interpret this result saying "the circumradius of $L$ has the steepest increase if we change the PQF $\mathrm{id}_d$ in such a way that the length of the longest edge $\operatorname{conv}\{\boldsymbol{0}, 2\boldsymbol{e}_1\}$ of $L$ has the largest increase". Now

$$\pi_{\mathcal{H}}(-g_{L,1}(\mathrm{id}_d)) = \alpha \operatorname{diag}(d-1, -1, \ldots, -1)$$

with $\alpha > 0$, which shows that

$$\pi_{\mathcal{H}}(-g_{L,1}(\mathrm{id}_d)) = \alpha \sum_{i=2}^{d} N_i.$$

Thus we can apply Theorem 5.47. □

We note that the existence of covering pessima shows in particular that $\Theta$ is not a *topological Morse function*, which is a major difference to the lattice packing density (see [**4**]).

## 5.7. Local covering maxima of subspaces and Minkowski's conjecture

Instead of considering local covering maxima with respect to $\mathcal{S}^d_{>0}$, we may ask more generally for local covering maxima with respect to subsets of $\mathcal{S}^d_{>0}$. It could happen, depending on the considered subset, that there exist infinitely many inequivalent local covering maxima. In case of a linear subspace $T \subseteq \mathcal{S}^d$, we can use the theory developed in Section 4.3. It enables us to enumerate all inequivalent local covering maxima with respect to $T \cap \mathcal{S}^d_{>0}$, in case there are only finitely many.

We explain below that due to a recent result of C.T. McMullen [**176**], this technique can (in principle) be used to computationally verify the so-called *Minkowski conjecture* in a given dimension.

**5.7.1. Minkowski's conjecture.** Let $N(\boldsymbol{x}) = |x_1 x_2 \cdots x_d|$ be the *norm function* on $\mathbb{R}^d$. Then the Minkowski conjecture (cf. [**97**]) can be stated as follows.

CONJECTURE 5.49. *Lattices $L \subset \mathbb{R}^d$ with* $\det L = 1$ *satisfy*

$$\sup_{\boldsymbol{x} \in \mathbb{R}^d} \inf_{\boldsymbol{y} \in L} N(\boldsymbol{x} - \boldsymbol{y}) \leq 2^{-d}$$

*and equality holds if and only if* $L = \operatorname{diag}(a_1, \ldots, a_d)\mathbb{Z}^d$ *with* $N(\boldsymbol{a}) = 1$.

The conjecture arose in the study of algebraic integers and yields a bound for their density in totally real number fields. Due to the inequality

$$N(\boldsymbol{x})^{1/d} \leq \sqrt{\tfrac{1}{d}(x_1^2 + \ldots + x_d^2)}$$

between the arithmetical and geometrical mean, it has been observed (cf. the survey [**12**]) that the conjecture follows if one verifies the following two assertions:

(i) **Well-roundedness:** For a lattice $L \subset \mathbb{R}^d$ there exists an $\boldsymbol{a} \in \mathbb{R}^d$ with $N(\boldsymbol{a}) = 1$, such that $L' = \operatorname{diag}(a_1, \cdots, a_d)L$ is *well-rounded*, that is, the minimal vectors of $L'$ span $\mathbb{R}^d$.

(ii) **Covering conjecture:** The covering density of any well-rounded lattice $L$ satisfies

$$\Theta(L) \leq \left(\tfrac{\sqrt{d}}{2}\right)^d \operatorname{vol}(B^d) = \Theta(\mathbb{Z}^d) \tag{5.21}$$

and equality holds if and only if $L$ is similar to $\mathbb{Z}^d$.

The Minkowski conjecture has subsequently been verified in dimensions $2, \ldots, 5$ by Minkowski himself [**182**], by Remak [**202**], Dyson [**97**] and Skubenko [**238, 237**] (see [**12**] a detailed account). Woods [**268**] proved that the covering conjecture is true in dimension 6 (cf. [**132**]). A major breakthrough was achieved by Curtis McMullen [**176**], who proved (i) for every dimension. Due to the result by Woods, the Minkowski conjecture has therefore been verified for dimensions $d \leq 6$ so far. Of course, the main focus is now on the covering conjecture. Just recently Hans-Gill, Raka and Sehmi [**131**] announced a proof of the covering conjecture (and hereby of Minkowski's conjecture) for $d = 7$.

Below we explain how the covering conjecture can (in principle) computationally be proven or disproven in a fixed dimension. This is due to the fact that it is possible to enumerate the finitely many local covering maxima with respect to the space of isometry classes of well-rounded lattices.

**5.7.2. Covering maxima among well-rounded lattices.** We use the language of PQFs. Recall from Section 2.3 the definition of the successive minima. The bases of well-rounded lattices up to orthogonal transformations are identified with

$$\mathcal{W}^d = \{Q \in \mathcal{S}^d_{>0} : \lambda_1(Q) = \ldots = \lambda_d(Q)\}.$$

The set $\mathcal{W}^d$ is a union of countably many closed polyhedral cones of dimension $\binom{d+1}{2} - (d-1)$. The group $\mathsf{GL}_d(\mathbb{Z})$ acts on $\mathcal{W}^d$ by $Q \mapsto U^t Q U$ and the isometry classes of well-rounded lattices are identified with the quotient $\mathcal{W}^d/\mathsf{GL}_d(\mathbb{Z})$.

THEOREM 5.50. *In a fixed dimension $d$, there exist only finitely many local covering maxima with respect to $\mathcal{W}^d/\mathsf{GL}_d(\mathbb{Z})$.*

PROOF. The set $\mathcal{W}^d$ is contained in a union of countably many linear subspaces $T$ of dimension $\binom{d+1}{2} - (d-1)$. So there is a cover of $\mathcal{W}^d$ by $T$-secondary cones. As in the proof of Theorem 5.42 we find that local lattice covering maxima within $T$ can only be attained by *$T$-extreme Delone polytopes*, that is, Delone polytopes for which $\overline{\boldsymbol{\Delta}(P, \mathbb{Z}^d)} \cap T$ is a ray. These rays are a subset of the set of $T$-rigid $T$-secondary cones. Thus in order to prove the theorem, it is sufficient to show that $\mathcal{W}^d$ contains only finitely many $T$-rigid $T$-secondary cones up to $\mathsf{GL}_d(\mathbb{Z})$-equivalence.

By Voronoi's first reduction theory (respectively the theory of Ryshkov polyhedra described in Chapter 3) we know that we can find a finite collection $\mathcal{C}$ of sets with $d$ linearly independent integral vectors such that every $T$-rigid $T$-secondary cone is $\mathsf{GL}_d(\mathbb{Z})$-equivalent to one in

$$\{Q \in \mathcal{S}^d_{>0} : \lambda(Q) = Q[\boldsymbol{s}_1] = \ldots = Q[\boldsymbol{s}_d], \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d\} \in \mathcal{C}\}.$$

For example, we may obtain $\mathcal{C}$ from a complete collection of $\mathsf{GL}_d(\mathbb{Z})$-inequivalent $(\binom{d+1}{2} - d)$-dimensional faces of a Ryshkov polyhedron. Each face corresponds to $d$ integral (not necessarily linearly independent) vectors $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d\} \subset \mathbb{Z}^d$ with $\lambda(Q) = Q[\boldsymbol{s}_1] = \ldots = Q[\boldsymbol{s}_d]$.

For a fixed choice of linearly independent $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d\} \subset \mathbb{Z}^d$,

$$T = \left\{Q \in \mathcal{S}^d : Q[\boldsymbol{s}_1] = \ldots = Q[\boldsymbol{s}_d]\right\}$$

is a linear subspace of $\mathcal{S}^d$. The polyhedral cone

$$\left\{Q \in \mathcal{S}^d_{>0} : \lambda(Q) = Q[\boldsymbol{s}_1] = \ldots = Q[\boldsymbol{s}_d]\right\} \tag{5.22}$$

is a subset of $T$. It is the conic hull of

$$\left\{Q \in \mathcal{S}^d_{>0} : \lambda(Q) = Q[\boldsymbol{s}_1] = \ldots = Q[\boldsymbol{s}_d] = 1\right\}. \tag{5.23}$$

If we show that (5.23) is compact, then (by the definition of a fundamental domain and the reduction theory of Voronoi) the set (5.23) and hence (5.22) has a non-empty intersection with only finitely many secondary cones $\boldsymbol{\Delta}(\mathcal{D})$. It follows that (5.22) contains only finitely many $T$-rigid $T$-secondary cones, which proves the theorem.

It remains to verify that (5.23) is compact. The set is closed because any convergent sequence $\{Q_i\}$ with $Q_i$ in (5.23) has its limit also in (5.23). In order to see that the set (5.23) is bounded it suffices to prove that the trace of contained PQFs $Q$ is bounded by some constant. For this we observe that the cross polytope

$$\operatorname{conv}\{\pm\boldsymbol{s}_1, \ldots, \pm\boldsymbol{s}_d\} \tag{5.24}$$

is full dimensional, and for a suitable constant $C > 0$,

$$C \cdot \operatorname{conv}\{\pm\boldsymbol{e}_1, \ldots, \pm\boldsymbol{e}_d\}$$

is contained in it. The ellipsoid $E(Q, 1) = \{\boldsymbol{x} \in \mathbb{R}^d : Q[\boldsymbol{x}] \leq 1\}$ contains (5.24). Thus $Q[C\boldsymbol{e}_i] = C^2 q_{ii} \leq 1$, for $i = 1, \ldots, d$, which yields $\operatorname{trace} Q \leq dC^{-2}$ and hereby finishes the proof. □

As a byproduct, the proof yields an algorithm to verify or disprove the covering conjecture in a given dimension. Due to the result of Curtis McMullen we can use the algorithm to verify (but not necessarily disprove) Minkowski's conjecture in a given dimension.

First we determine a finite collection $\mathcal{C}$ as in the proof. Then for each choice of linearly independent $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_d\} \in \mathcal{C}$ we find the $T$-rigid $T$-secondary cones in (5.22). For each of these cones (which are in fact rays) we compute the covering density attained on them. If one of these densities is as large or even larger than $\left(\frac{\sqrt{d}}{2}\right)^d \operatorname{vol}(B^d) = \Theta(\mathrm{id}_d)$, the covering conjecture is false for $d$; otherwise the covering and the Minkowski conjecture are true.

APPENDIX A

# Polyhedral Representation Conversion under Symmetries

Much of the computations described in this book heavily rely on polyhedral computations. On the one hand they are the core of Voronoi's and Minkowski's theories; in a way their theories even stimulated the development of the theory of polyhedra in their times. On the other hand, the computational tools for practical polyhedral computations, such as linear programming are today developed to an extend, which allows to treat even very large problems mathematical rigorously by computers.

One key tool accompanying this book is the *representation conversion of polyhedra with large symmetries*, to be explained below. Over several years, while we were working on different problems involving such computations, Mathieu Dutour Sikirić has developed sophisticated `GAP` [**276**] programs that made some computational results described in this book possible (see also [**91**], [**92**] and [**93**]). His implementations are available from his web page [**283**].

## A.1. Convex polyhedra

In this section, we give a brief introduction to some basic concepts and terminology of (convex) polyhedra. For more details on polyhedra, we refer to the books [**126**], [**177**], [**233**], [**270**].

Given the vector space $\mathbb{R}^d$, denote by $(\mathbb{R}^d)^*$ its dual vector space, i.e. the vector space of linear functionals on $\mathbb{R}^d$. A *convex polyhedron* $\mathcal{P} \subseteq \mathbb{R}^d$ can be defined by a finite set of linear inequalities

$$\mathcal{P} = \{x \in \mathbb{R}^d : f_i(x) \geq b_i, i = 1, \ldots, m\}$$

with $f_i \in (\mathbb{R}^d)^*$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$. If the number of inequalities $m$ in the description is minimum, we speak of a non-redundant description. The dimension $\dim \mathcal{P}$ of $\mathcal{P}$ is the dimension of the smallest affine subspace containing it. Under the assumption that $\mathcal{P}$ is full-dimensional, i.e. $\dim \mathcal{P} = d$, every inequality $i$ of a non-redundant description defines a *facet* $\{x \in \mathcal{P} : f_i(x) = b_i\}$ of $\mathcal{P}$, which is a $(d-1)$-dimensional convex polyhedron contained in the boundary of $\mathcal{P}$.

By the Farkas-Minkowski-Weyl Theorem (see e.g. [**233**, Corollary 7.1a]), $\mathcal{P}$ can also be described by a finite set of generators:

$$\begin{aligned} \mathcal{P} &= \operatorname{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} + \operatorname{cone}\{\boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_n\} \\ &= \{\sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i : \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\} \end{aligned}$$

where $\boldsymbol{v}_i \in \mathbb{R}^d$ for $i = 1, \ldots, n$. If the number of generators is minimum, the description is again called *non-redundant*. In the non-redundant case, the generators

$\boldsymbol{v}_i$, $i = 1, \ldots, k$, are called *vertices* and $\mathbb{R}_{\geq 0}\boldsymbol{v}_i$, $i = k+1, \ldots, n$, are the *extreme rays* of $\mathcal{P}$. In case $\mathcal{P}$ is bounded we have $n = k$ and we speak of a *convex polytope*.

The *representation conversion* from a minimal set of generators into a minimal set of linear functionals (or vice versa) is called the *dual description problem*. For certain classes of polyhedra efficient methods are known, but there is no approach known which efficiently solves the problem in general. Programs like `cdd` [**275**], `lrs` [**278**], `pd` [**282**], `porta` [**284**] (or `polymake` [**115**] either relying on some of the others or using its own method closely related to `cdd`) allow conversion of the representation of a polyhedron. Since the programs are implementations of quite different methods, their efficiency may vary tremendously on a given example. For complexity theoretic results we refer the interested reader to [**96**] and [**152**].

By using homogeneous coordinates, the general inhomogeneous problem stated above can be reduced to the homogeneous one where $b_i = 0$ and $k = 0$. For example, we embed $\mathcal{P} \in \mathbb{R}^d$ in the hyperplane $x_{d+1} = 1$ in $\mathbb{R}^{d+1}$ and consider the closure of its *conic hull*. The so obtained polyhedron $\mathcal{P}' = \operatorname{cone}\{\boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n\}$, with $\boldsymbol{v}'_i = (\boldsymbol{v}_i, 1)$ for $i = 1, \ldots, k$ and $\boldsymbol{v}'_i = (\boldsymbol{v}_i, 0)$ for $i = k+1, \ldots n$, is referred to as *polyhedral cone*.

By duality, the problem of converting a description by homogeneous inequalities into a description by extreme rays is equivalent to the opposite conversion problem. So for simplicity we assume from now on that $\mathcal{P} \subseteq \mathbb{R}^d$ is a *polyhedral cone* given by a minimal (non-redundant) set of generators (extreme rays). If

$$\mathcal{P} = \operatorname{cone}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$$

we say that $\mathcal{P}$ is generated by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^d$.

We want to find a minimal set $\{f_1, \ldots, f_m\} \subset (\mathbb{R}^d)^*$ with

$$\mathcal{P} = \{x \in \mathbb{R}^d : f_i(x) \geq 0, i = 1, \ldots, m\}.$$

By choosing a suitable projection, it is possible to reduce the problem further to the case where $\mathcal{P}$ is full-dimensional and does not contain any non-trivial linear subspaces. For example, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^d$ span a $k$-dimensional linear subspace, we may just choose (project onto) $k$ independent coordinates. The appropriate projection (i.e. equations of the linearity space) can be found efficiently via Gaussian elimination. In other words, without loss of generality we may assume the $\boldsymbol{v}_i$ span $\mathbb{R}^d$ ($\mathcal{P}$ is full-dimensional) and the linear inequalities $f_i$ span $(\mathbb{R}^d)^*$ ($\mathcal{P}$ is pointed).

A *face* of $\mathcal{P}$ is a set $\{x \in \mathcal{P} : f(x) = 0\}$ where $f$ is an element of

$$\mathcal{P}^* = \{f \in (\mathbb{R}^d)^* : f(x) \geq 0 \text{ for all } x \in \mathcal{P}\},$$

the polyhedral cone *dual* to $\mathcal{P}$. Note that $(\mathcal{P}^*)^* = \mathcal{P}$. The faces of a pointed polyhedral cone are themselves pointed polyhedral cones. We speak of a $k$-face, if its dimension is $k$. The faces form a (combinatorial) lattice ordered by inclusion, the *face lattice* of $\mathcal{P}$. The rank of a face in the lattice is given by its dimension. Each face is generated by a subset of the generators of $\mathcal{P}$ and therefore it is uniquely identified by some subset of $\{1, \ldots, n\}$. In particular, the 0-dimensional face is identified with the empty set $\emptyset$ and $\mathcal{P}$ itself with the full index set $\{1, \ldots, n\}$. All other faces of $\mathcal{P}$ are identified with some strict, non-empty subset $F \subset \{1, \ldots, n\}$.

We write $F \lhd F'$ for two faces of $\mathcal{P}$ with $F \subset F'$ and $\dim F = \dim F' - 1$. Two $k$-faces of $\mathcal{P}$ are said to be *adjacent*, if they share a $(k-1)$- and a $(k+1)$-face. In particular, two extreme rays (1-faces) are adjacent, if they generate a common 2-face and two facets are adjacent, if they share a common $(d-2)$-face (a *ridge*).

By the properties of a lattice, for two faces $F_1$ and $F_2$, not necessarily of the same dimension, there is always a unique largest common face $F = F_1 \cap F_2$ contained in them, and a unique smallest face $F'$, containing both. Any sub-lattice $[F : F']$ of the face lattice consisting of all faces containing $F$ and contained in $F'$ is known to be isomorphic to the face lattice of some pointed polyhedral cone of dimension $\dim F' - \dim F$. Note that this is a special feature of face lattices of polyhedra, which is not true for general lattices. In particular the *diamond property* holds for face lattices: Every sub-lattice of rank 2 carries the combinatorics of a 2-dimensional polyhedral cone, namely a face $F$ of rank $k-1$, a face $F'$ of rank $k+1$ and two faces $F_1$ and $F_2$ of rank $k$. The rank 2 sub-lattices are in one-to-one correspondence to pairs of adjacent faces.

A *polyhedral complex* $\Delta$ is a set of polyhedral cones (the *cells* of $\Delta$), such that for each cell of $\Delta$, its faces are also cells of $\Delta$, and such that the following *intersection property* holds:

**(IP)** For all $P_i, P_j \in \Delta$, $P_i \cap P_j$ is a (possibly empty) face of both $P_i$ and $P_j$.

The facets of a polyhedral cone $\mathcal{P}$ form a polyhedral complex, the *boundary complex* of $\mathcal{P}$. Polyhedral complex $\Delta'$ is a *subdivision* of polyhedral complex $\Delta$ if every cell of $\Delta$ is the union of cells in $\Delta'$ and every cell of $\Delta'$ is contained in some cell of $\Delta$. A subdivision is called a *triangulation* if every cell is a simplicial cone. By the homogeneous embedding of polytopes discussed above, we may equally discuss boundary complexes, subdivisions, and triangulations for polytopes.

## A.2. Polyhedral symmetries

**A.2.1. Groups acting on polyhedra.** In this book we are especially interested in the case where some non-trivial group acts on the given polyhedron $\mathcal{P}$. The *combinatorial automorphism group* of $\mathcal{P}$, generated by $n$ extreme rays, is the subgroup of $\mathrm{Sym}(n)$ of all permutations (acting on ray indices $\{1, \ldots, n\}$) which preserve the whole face lattice of $\mathcal{P}$. Thus the combinatorial automorphism group acts not only on the generating rays, but also on the set of facets (inequalities), and more generally on the sets of faces of a given dimension. Moreover, it preserves the inclusion relation among faces. It is known that the combinatorial automorphism group of $\mathcal{P}$, can be computed from the incidence relations between extreme rays and facets (see [**148**]).

Given generators $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^d$ of a polyhedral cone $\mathcal{P}$ and a subgroup $G$ of the combinatorial automorphism group of $\mathcal{P}$, we want to obtain the facets of $\mathcal{P}$ up to symmetry, that is, one representative $F \subset \{1, \ldots, n\}$ from each orbit under the action of $G$.

If no group or only a small group is given, we usually still want to exploit as much symmetries of the given polyhedron as possible in the representation conversion. That is, ideally we would like to compute the full combinatorial automorphism group. However, we do not know how to compute it without computing the facets, which is precisely the problem we want to solve. So we have to settle for a compromise and work with a more restricted type of automorphism. Note though, that one may obtain the full combinatorial automorphism group of $\mathcal{P}$, after the facets have been computed

In many cases the combinatorial automorphism group (or some nontrivial subgroup) may reflect geometric symmetries, for example if it has a representation as a subgroup of $\mathsf{GL}_d(\mathbb{R})$ acting naturally on $\mathbb{R}^d$ and the polyhedral cone $\mathcal{P}$. The group of all matrices in $\mathsf{GL}_d(\mathbb{R})$ preserving $\mathcal{P}$ is called *linear automorphism group* of $\mathcal{P}$. Since a linear automorphism permutes the set of extreme rays $\{\mathbb{R}_{\geq 0}\boldsymbol{v}_1, \ldots, \mathbb{R}_{\geq 0}\boldsymbol{v}_n\}$, we naturally obtain a representation as a permutation group $G \leq \mathrm{Sym}(n)$. It is important to note here that although the induced permutation group is finite, the linear automorphism group of $\mathcal{P}$ is not necessarily so, and it can be quite awkward. Think for example of the quadrant in $\mathbb{R}^2$ generated by the two non-negative coordinate axes. The induced permutation group is Sym(2), its linear automorphism group however is

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R}_{>0} \right\}.$$

As explained in Section A.1 we may limit our discussion to the special case of a full-dimensional, pointed polyhedral cone $\mathcal{P}$ generated by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^d$. The elements $A \in \mathsf{GL}_d(\mathbb{R})$ of the linear automorphism group of $\mathcal{P}$ satisfy $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_{\sigma(i)}$, where $\sigma \in \mathrm{Sym}(n)$ is an induced permutation and $\lambda_i > 0$. Note, in case $\mathcal{P}$ is the homogenization of a $(d-1)$-dimensional polyhedron $\mathcal{P}'$, the linear automorphism group of $\mathcal{P}$ corresponds to the so-called *projective automorphism group* of $\mathcal{P}'$, that is, the set of all projective maps preserving $\mathcal{P}'$. We are not aware of any practical algorithm to compute linear (or projective) automorphism groups or to decide if two polyhedral cones are linear (or projective) isomorphic.

**A.2.2. Restricted isomorphisms.** A *restricted isomorphism* of two full-dimensional vector families $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $V' = \{\boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_n\}$ in $\mathbb{R}^d$ is given by a matrix $A \in \mathsf{GL}_d(\mathbb{R})$ such that there exists a permutation $\sigma$ satisfying $A\boldsymbol{v}_i = \boldsymbol{v}'_{\sigma(i)}$ for $i = 1, \ldots, n$. A *restricted automorphism* of a vector family is a restricted isomorphism of $V$ with itself.

We speak of a restricted isomorphism between two polyhedral cones generated by the two vector families $V$ and $V'$, if it is a restricted isomorphism between $V$ and $V'$. Note though that the definition of restricted isomorphisms for polyhedral cones is strongly depending on the choice of generators. Since such generators are only unique up to any positive factor, the choice of generators is very crucial. In practice the situation is usually not so bad as there is often a natural choice for the generators.

The big advantage of restricted isomorphisms is that we can compute them by obtaining the graph isomorphisms for an edge labeled graph. Given a vector family $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset \mathbb{R}^d$ spanning $\mathbb{R}^d$ we define the positive definite matrix

$$Q = \sum_{i=1}^{n} \boldsymbol{v}_i \boldsymbol{v}_i^t. \tag{A.1}$$

Let the graph $G(V)$ be the complete graph with vertices $\boldsymbol{v}_i$ and edge weights $c_{ij} = v_i^t Q^{-1} \boldsymbol{v}_j$. Then the following holds:

Proposition A.1. *Let $V, V' \subset \mathbb{R}^d$ be two finite vector families. Then every isomorphism of the edge labeled graphs $G(V)$ and $G(V')$ yields a restricted isomorphism of $V$ and $V'$ and vice versa.*

For a proof of the proposition, we refer to [**45**]. In practice the popular and very nice program `nauty` [**281**] by McKay can be used to check for graph isomorphisms or compute the group of automorphisms. Note however, that the current version only takes vertex labeled graphs as input. We therefore have to transform our edge labelled graph to a somewhat larger vertex labelled graph that preserves the automorphism group (see for example [**281**], User's Guide, Version 2.4, p.25).

## A.3. Orbits of faces

**A.3.1. Dealing with orbits.** In order to generate facets (or more generally faces) of a polyhedron up to symmetries, it is necessary to deal with orbits of faces. In this section we briefly indicate what the basic tasks are that we have to accomplish and how these can be approached.

As before, we assume that a polyhedral cone $\mathcal{P}$ is given by a set of generators $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$ which define the extreme rays of $\mathcal{P}$. Each face is represented by a subset of $\{1, \dots, n\}$ which corresponds to the indices of generators incident to the face. We assume $G \leq \mathrm{Sym}(n)$ is some subgroup of the combinatorial automorphism group of $\mathcal{P}$, hence a permutation group acting not only on the set of indices $\{1, \dots, n\}$ (respectively rays), but also on the whole face lattice of $\mathcal{P}$. In particular, the dimension of a face and inclusion between faces are preserved by every group element.

In the most general form, the problem we have to solve is the following: Given two subgroups $G_1$ and $G_2$ of $\mathrm{Sym}(n)$ and a list $L_1$ of $G_1$-inequivalent faces, we need to be able to obtain a list $L_2$ of $G_2$-inequivalent faces.

For example, if $G_1 = \{\mathrm{id}_d\}$ is trivial, the list $L_1 = \{F_1, \dots, F_k\}$ simply is a list of pairwise unequal faces and we may obtain $L_2$ by testing for $G_2$-equivalence: Starting with $L_2 = \{F_1\}$ and then subsequently adding $F_i$ for $i = 2, \dots, n$ to $L_2$, if it is not $G_2$-equivalent to any element in $L_2$. Clearly, this *orbit fusion* can be applied whenever $G_1 \leq G_2$.

In case $G_2 < G_1$ it is necessary to "break some symmetry" and *split (factorize) orbits*. This can be done with the *double coset decomposition*, described in Section A.3.2. Note, if neither $G_1 \leq G_2$ nor $G_2 < G_1$, one could in principle convert the $G_1$-orbits into $G_2$-orbits in two steps, by either using the intersection group $G = G_1 \cap G_2$ or the group $G = \langle G_1, G_2 \rangle$ generated by $G_1$ and $G_2$ and obtaining $G$-orbits in an intermediate step.

Orbit fusing and splitting is a typical and essential task when generating discrete structures up to isomorphism (see for example [**48**], [**127**], [**149**], [**150**]).

**A.3.2. Fusing orbits, equivalence tests and canonical representatives.** A common task is to decide whether or not two faces are $G$-equivalent, that is, whether or not the corresponding subsets of $\{1, \dots, n\}$ lie in the same orbit under the action of $G$.

The dimension of a face (rank in the face lattice) and its cardinality are quickly testable invariants. Other invariants can easily be obtained, for example by looking at the action of $G$ on pairs, triples, or other $k$-tuples of indices (generators), respectively on lower dimensional faces. The number of elements from each such orbit included in a face is a $G$-invariant. Unfortunately, there is no clear rule how many such sets have to be chosen, so one has to rely on heuristics.

If we only consider restricted isomorphisms, metric invariants can be used as well, for example the set of pairwise inner products between generators (see [**45**]).

When all invariants are satisfied, then group computations must be done. For small groups it is possible to simply generate the whole orbit of a face. If the size of available memory is a problem, we may just keep a *canonical representative* for the orbit, for example the lexicographical minimum (when viewed as a subset of $\{1, \ldots, n\}$). These can be found for example by a backtrack method (see Section A.3.4).

**A.3.3. Breaking symmetry by splitting orbits using double cosets.** Assume $G_2 < G_1 \leq \mathrm{Sym}(n)$ and that a list of $G_1$-orbits is given. In order to obtain a list of $G_2$-orbits, we may split each orbit $G_1F$, with a representative face $F$, by the well known *double coset decomposition* (cf. for example in [**48**] and [**150**]): The group $G_1$ can be decomposed into *double cosets*

$$G_1 = \bigcup_{i=1}^{r} G_2 g_i \,\mathrm{Stab}(G_1, F)$$

where $g_1, \ldots, g_r$ are elements of $G_1$. Then the orbit $G_1F$ is decomposed into

$$G_1F = \bigcup_{i=1}^{r} G_2 g_i F,$$

hence into $r$ orbits $G_2F_i$ with $F_i = g_iF$.

Note that the more straightforward algorithm of generating the full orbit or of computing a decomposition of $G_1$ into cosets $G_2g_i$ is slower and/or requires more memory (see [**150**]).

**A.3.4. Data structures and implementation issues.** The fundamental data structures used to work with permutation groups are *bases and strong generating sets* (BSGS) (see for example [**231**], [**138**]). Based on them a backtrack search on cosets of a point stabilizer chain can be used to obtain canonical representatives, to decide on (non-)equivalence as well as to obtain stabilizers of subsets of $\{1, \ldots, n\}$ (faces). For details we refer to [**231**] and [**149**]. An elaborate version is the *partition backtrack* introduced by Leon [**168**] (cf. [**169**], [**231**]). These methods are known to work quite well in practice, although from a complexity point of view the problems are known to be difficult. That is, there is no (worst-case) polynomial time algorithm known to solve these problems. Even worse, it is somewhat unlikely that there exist polynomial time algorithms, since the *graph isomorphism problem*, not known to be in $P$, is reducible to them in polynomial time (cf. [**171**]).

The computer algebra system `GAP` [**276**] provides functions for generation of full orbits (ORBIT) stabilizer computations (STABILIZER) and equivalence tests (REPRESENTATIVEACTION). In all the used algorithms (backtrack searches) the number of group generators of the given group is very important. So from a practical point of view one should try to (heuristically) obtain a small set of group generators. `GAP` provides a function for this task as well (SMALLGENERATINGSET).

In the case where we have special knowledge of the group $G$ or its representation, it might be much easier to obtain canonical representatives or to compute stabilizers of faces. For example, we may have a situation where the symmetric group $\mathrm{Sym}(n)$ acts on $n$ elements (see [**2**] and [**53**]). Another example where the action of the symmetric group is used is described in [**81**]. Typically, polyhedra arising in Combinatorial Optimization are convex hulls of (0/1)-vectors, where each coordinate (variable) represents an edge of a complete directed or undirected graph with $n$ vertices on which $\mathrm{Sym}(n)$ acts. In [**56**] a method for obtaining canonical representatives in this situation is described.

## A.4. Decomposition methods

In this section we describe two methods which reduce the facet generation problem under symmetries to a number of smaller instances of facet generation problems. In contrast to the original problem, solving the smaller problems (for subcones) may be feasible for available software, such as `cdd` or `lrs`. These techniques have been proven to be successful in practice, in cases where standard approaches failed.

We mainly distinguish two approaches, the *Incidence Decomposition Method* (see Section A.4.1) and the *Adjacency Decomposition Method* (see Section A.4.2). Both methods are reasonably natural and have been used separately by different authors.

**A.4.1. Incidence Decomposition Method.** The Incidence Decomposition Method reduces the problem of facet generation to a number of smaller problems, in which we generate facets that are incident to some extreme rays. As before let $\mathcal{P}$ be a polyhedral cone in $\mathbb{R}^d$, generated by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Assume $G \leq \mathrm{Sym}(n)$ is some permutation group acting on the face lattice of $\mathcal{P}$. The set of extreme rays (indices) falls into orbits under the action of $G$. For each orbit we consider a representative $r_i$ (index $i$) and generate a list of $G$-inequivalent facets of $\mathcal{P}$ incident to it. Then, in a post-processing step the lists of facets obtained in this way are merged to a list of $G$-inequivalent facets of $\mathcal{P}$ (see Section A.3). Since every $G$-orbit of facets of $\mathcal{P}$ contains a facet which is incident to one of the chosen representatives, the resulting list is complete.

**Input:** $n$ extreme rays of a polyhedral cone $\mathcal{P}$ and a group $G \leq \mathrm{Sym}(n)$ acting on $\mathcal{P}$'s face lattice.
**Output:** complete set $\mathcal{F}$ of $G$-inequivalent facets of $\mathcal{P}$.
$\mathcal{F} \leftarrow \emptyset$.
$\mathcal{R} \leftarrow$ complete set of $G$-inequivalent extreme rays of $\mathcal{P}$.
**for** $r \in \mathcal{R}$ **do**
  $\mathcal{F}_r \leftarrow$ facets of $\mathcal{P}$ incident to $r$.
  **for** $F \in \mathcal{F}_r$ **do**
    **if** $F$ is $G$-inequivalent to facets in $\mathcal{F}$ **then**
      $\mathcal{F} \leftarrow \mathcal{F} \cup \{F\}$.
    **end if**
  **end for**
**end for**

ALGORITHM 8. Incidence Decomposition Method.

The main computational gain comes from the following: when we compute the facets that are incident to a given ray $r_i$, we may not have to consider all $n$ extreme rays of $\mathcal{P}$, because some of the rays may not be incident to facets which are incident to $r_i$. These are exactly the extreme rays $r_j = \mathbb{R}_{\geq 0}\boldsymbol{v}_j$ which give redundant inequalities of the polyhedron

$$P_i^* = \{f \in (\mathbb{R}^d)^* : f(\boldsymbol{v}_j) \geq 0, j = 1, \ldots, n \text{ and } f(\boldsymbol{v}_i) = 0\}.$$

So, for each of the $n-1$ rays with index in $\{1,\ldots,n\} \setminus \{i\}$ we may solve a linear program in $(\mathbb{R}^d)^*$ to decide redundancy. As an outcome we obtain a list of $n' < n$ rays needed in the definition of $P_i^*$. The smaller $n'$, the bigger the computational gain. Note that $P_i^*$ is of dimension $d-1$. Its dual contains the line $\{\lambda \boldsymbol{v}_i : \lambda \in \mathbb{R}\}$ and we may project it (or at least think of it as projected) down along this line to a $d-1$ dimensional polyhedral cone.

So the problem of enumerating facets incident to a given ray is a facet enumeration problem, but in one lower dimension and with fewer inequalities. Some of the lower dimensional subproblems may still be too difficult; in this case we may apply the method recursively. We come back to this in Section A.4.3.

The Incidence Decomposition Method has been used in [**117**] for finding the vertices of the metric polytope $\mathrm{MET}_7$. The method was also introduced in [**55**] along with the Adjacency Decomposition Method but it was found to be less competitive for their application. The Incidence Decomposition Method was also discussed by Fukuda and Prodon [**111**].

**A.4.2. Adjacency Decomposition Method.** As with the Incidence Decomposition Method, the Adjacency Decomposition Method is a reasonably natural method for computing the facets of a polytope up to symmetries. So it is no wonder that the method was discovered several times, for example, in [**143**] as “algorithm de l'explorateur”, in [**55**] as “adjacency decomposition method” and in [**80**] as “subpolytope algorithm”. The adjacency decomposition scheme is also implicit in polyhedral decomposition schemes, i.e. when a space is decomposed as an union of polyhedral cones, as we have seen in Chapters 3 and 4.

Rather than focusing on the incidence of (orbits of) facets to extreme rays as the Incidence Decomposition Method does, the Adjacency Decomposition Methods focusses on the incidence of facets with other facets. Starting from a (set of) initial $G$-inequivalent facets, it traverses the adjacency graph of facet orbits. The initial facet(s) may be obtained by suitable linear programs, or in strongly polynomial time using the methods described in [**46**].

The facet $F'$ of $\mathcal{P}$ with $F \cap F' = H$ for a given ridge $H \lhd F$ can be found by a *gift-wrapping step* (cf. [**54, 246**]). Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be the generators of $\mathcal{P}$'s extreme rays. The defining inequality $f \in (\mathbb{R}^d)^*$ of the facet $F'$ should satisfy $f(\boldsymbol{v}_i) = 0$ for all generators $\boldsymbol{v}_i \in H$. The vector space of such functions has dimension 2. Let us select a basis $\{f_1, f_2\}$ of it. If $f = \alpha_1 f_1 + \alpha_2 f_2$ is the defining inequality of $F$, $H$ or $F'$, then $f(\boldsymbol{v}_i) \geq 0$ for all $i \in \{1, \ldots, n\}$. This translates into a set of linear inequalities on $\alpha_1, \alpha_2$ defining a 2-dimensional pointed polyhedral cone. One finds easily its two generators $(\alpha_1^i, \alpha_2^i)_{1 \leq i \leq 2}$. The corresponding inequalities $f_i = \alpha_1^i f_1 + \alpha_2^i f_2 \in (\mathbb{R}^d)^*$ define the two adjacent facets $F$ and $F'$ of $\mathcal{P}$. In the special case where no $d+1$ extreme rays lie on a hyperplane, the gift wrapping step corresponds to a simplex pivot (see [**57**]).

A nice feature of the Adjacency Decomposition Method is that the adjacencies of the most symmetric facets, which are usually the most difficult to treat, may not have to be considered. This is due to the following well known result, known as *Balinski's theorem* (see e.g. [**270**], Theorem 3.14):

THEOREM A.2. ([**7**]) *Let $\mathcal{P}$ be a $d$-dimensional, pointed polyhedral cone. Let $G$ be the undirected graph whose vertices are the facets of $\mathcal{P}$ and whose edges are*

**Input:** $n$ extreme rays of a polyhedral cone $\mathcal{P}$ and a group $G \leq \mathrm{Sym}(n)$ acting on $\mathcal{P}$'s face lattice.
**Output:** complete set $\mathcal{F}$ of $G$-inequivalent facets of $\mathcal{P}$.

```
T ← {F} with F a facet of P.
F ← ∅.
while there is an F ∈ T do
   F ← F ∪ {F}.
   T ← T \ {F}.
   H ← facets of F.
   for H ∈ H do
      find adjacent facet F′ of F in P with H = F ∩ F′.
      if F′ is G-inequivalent to all facets in F ∪ T then
         T ← T ∪ {F′}.
      end if
   end for
end while
```

ALGORITHM 9. Adjacency Decomposition Method.

*the ridges of* $\mathcal{P}$. *Two vertices* $F_1, F_2$ *are connected by an edge* $E$ *if* $F_1 \cap F_2 = E$. *Then, the graph* $G$ *is* $(d-1)$*-connected, i.e. removal of any* $d-2$ *vertices leaves it connected.*

Using Balinski's theorem, we know that if the number of facets in unfinished orbits (i.e. those whose neighbors are not known) is less than $d-1$, then any $G$-inequivalent facet in the unfinished set must be adjacent to some already completely treated facet. In practice this simple criterion can be extremely useful.

**A.4.3. Recursion.** Both, the Incidence Decomposition Method and the Adjacency Decomposition Method reduce the facet enumeration problem for a $d$-dimensional polyhedral cone $\mathcal{P}$ to a number of facet enumeration problems for cones in dimension $d-1$. These lower dimensional problems may be too difficult to treat with a standard method as well and therefore we might apply the Incidence or Adjacency Decomposition Method to these lower dimensional problems recursively. So we may speak of the *Recursive Incidence Decomposition Method*, the *Recursive Adjacency Decomposition Method* and the *Recursive Decomposition Method*, if a mixture of both is applied.

The Recursive Incidence Decomposition Method has successfully been used for computing the vertices of the metric polytope $\mathrm{MET}_8$ in [**79**], whereas the Recursive Adjacency Decomposition Method has successfully been used in [**143**] and [**91**]. To the best of our knowledge, and to our surprise, a recursive mixture of both has not been used so far.

The crucial steps for the recursion are "$\mathcal{F}_r \leftarrow$ facets of $\mathcal{P}$ incident to $r$" (where $r$ is some extreme ray) for the Incidence Method (see Section A.4.1) and "$\mathcal{F} \leftarrow$ facets of $F$" (where $F$ is some facet) for the Adjacency Decomposition Method (see Section A.4.2). In both cases the problem is to obtain a list of facets for a $(d-1)$-dimensional polyhedral cone $F$, whose extreme rays are given. Again we can exploit symmetries. As a result of a call of the Incidence or Adjacency Decomposition Method for $F$, we obtain a list of $G_F$-inequivalent facets of $F$ (ridges of $\mathcal{P}$), where

$G_F$ is some group acting on the face lattice of $F$ which we have to provide. In a post processing step we then have to obtain a list of $G_{\mathcal{P}}$-inequivalent (respectively $\mathrm{Stab}(G_{\mathcal{P}}, F)$-inequivalent) facets of $F$ (ridges of $P$) out of it.

Note though, that the groups $G_F$ and $G_{\mathcal{P}}$ may be unrelated, that is, the elements of $G_F$ do not have to be elements of $G_{\mathcal{P}}$ and vice versa. Think of examples where $\mathcal{P}$ is a polyhedral cone without any symmetries, but with a facet having some symmetries. Vice versa, since we did not assume that $G_F$ is the full (combinatorial) symmetry group, not even the stabilizer $\mathrm{Stab}(G_{\mathcal{P}}, F)$ of $F$ in $G_{\mathcal{P}}$ has to be a subgroup of $G_F$. In the latter case we may simply replace $G_F$ by the group generated by $\mathrm{Stab}(G_{\mathcal{P}}, F)$ and $G_F$, so that we assume $\mathrm{Stab}(G_{\mathcal{P}}, F) \leq G_F$. In this way we possibly enlarge the group and speed up the computations. Moreover, we are able to use the double coset decomposition (see Section A.3.3) to split each $G_F$-orbit $G_F F'$ of facets of $F$ into a finite number of $\mathrm{Stab}(G_{\mathcal{P}}, F)$-orbits $\mathrm{Stab}(G_{\mathcal{P}}, F) g_i F'$, where the $g_i \in G_F$ represent the double cosets.

Using the described decomposition methods recursively it might happen that we compute the facets of some sub-cones several times. For example if a face $F$ satisfies $F \lhd F_1 \lhd \mathcal{P}$ for a facet $F_1$ of a polyhedral cone $\mathcal{P}$, then there is exactly one other facet $F_2$ such that $F \lhd F_2 \lhd \mathcal{P}$. Hence, if we apply the Adjacency Decomposition Method to $F_1$ and $F_2$, then we have to compute the dual description of $F$ two times. Clearly, the number of such repetitions increases as the recursion depth increases. Moreover, equivalent sub-cones may occur in different parts of the face lattice. (Recall that any sub-lattice of the face lattice is the face lattice of some polyhedral cone.) To handle this, we propose to use a *banking system*. That is, given a difficult sub-cone $\mathcal{P}$ of which the facets have been computed up to symmetries, we store with $\mathcal{P}$ (represented by generators) its group of restricted automorphisms and a representative for each facet orbit.

What about the recursion depth needed to practically solve a given polyhedral conversion problem? If we split the problem into subproblems with either decomposition method then some of the subproblems may be easy to solve and may not require a recursive treatment, others might be impossible to treat without such. The implementation [**283**] allows one to provide a heuristic function to choose whether to recursively apply the Adjacency Decomposition Method, but choosing the right level of recursion requires some trial and error. From our computational experience with this code the *incidence number* of a face, that is, the number of extreme rays contained in it, gives a good measure for how difficult the representation conversion is for it. We therefore propose to treat the faces with low incidence numbers first and as much as possible without recursion. For subproblems to be potentially solved with pivoting methods (possibly up to symmetry as below), the "probing feature" of `lrs` can be used to test for expected difficulties of a subproblem.

Finally, let us remark, that parallelization is a possible way to speed up the (recursive) Decomposition Methods as well (see [**56**] and [**81**]).

APPENDIX B

# Possible Future Projects

Clearly, throughout such a book many interesting questions remain unanswered. In this appendix we collect some possible future projects. Most of them could become part or starting points for student projects or theses. We believe that the problems marked by $*$ are very challenging and not of this category. Note also that some of the problems require massive computational resources, unless one uses essential new theoretical insights. The list is in chronological order, that is, in order of their context occurring in the book.

$*$ Section 1.2: Show that the Leech lattice and the root lattice $\mathsf{E}_8$ attain $\delta_d^*$ for $d = 24$ and $d = 8$.

- Section 1.3: Show that Coxeter lattices aside of $\mathsf{A}_d^*$ do not give (locally) optimal lattice sphere coverings.
- Section 1.3: Obtain exact coordinates for the conjecturally optimal covering lattice $L_6^c$.

$*$ Section 1.3: Prove that the lattice $L_6^c$ gives the optimal lattice covering in dimension 6.

$*$ Section 1.3: Solve the "Kepler analog" for sphere coverings. That is, show that the *bcc-lattice* $\mathsf{A}_3^*$ attains $\Theta_3^*$.

$*$ Section 1.4: Decide whether or not there exists a dimension $d$ with $\gamma_d \geq 2$.

- Section 1.4: Prove or disprove that the root lattice $\mathsf{E}_8$ gives a locally optimal lattice packing-covering.
- Section 2.2: Classify the rays up to symmetry of $\mathcal{M}_d$ and/or $\mathcal{M}_d^+$ for $d = 6$ (or even larger $d$).
- Section 2.3: Prove our conjecture on the relation between successive minima and the length of basis vectors in a Minkowski reduced basis. Use for example computer assistance to verify it for dimension 6 and 7
- Section 2.4: Study multidimensional continued fraction expansions based on explicit reduction domains for Voronoi's theories.

$*$ Section 3.1: Classify all 9-dimensional perfect and extreme lattices and hereby solve the 9-dimensional lattice sphere packing problem.

- in Section 3.1: Computationally obtain explicit reduction domains from the known perfect forms.

- Section 3.2: Further analyze the best known packing lattices which are not strongly eutactic. Find out if some of them can be locally improved to yield denser periodic sphere packings.
- Section 3.3: Construct examples of $m$-extreme periodic point sets that are not $m'$-extreme.
- Section 3.4: Search for non-lattice packings (for example in dimensions $d \leq 10$) using the theory of $\boldsymbol{t}$-perfect forms and a Newton like method to "jump" and improve on $\boldsymbol{t}$.
- Section 3.4: Use the $T$-theory ($G$-theory) to systematically search for good sphere packing lattices and periodic sets in higher dimensions.
- Section 3.4: Use the $T$-theory to show that the Coxeter-Todd lattice is the unique optimal "perfect Eisenstein lattice" in dimension 12.
- Section 4.1: Classify all 6-dimensional lattice Delone triangulations.
- Section 4.2: Search for non-lattice coverings and packing-coverings in dimensions $d \leq 5$. Use for example the theory of $\boldsymbol{t}$-perfect forms and a Newton like method to "jump" and improve on $\boldsymbol{t}$.
- Section 4.3: Use the $T$-theory ($G$-theory) to systematically search for good sphere covering lattices and periodic sets in higher dimensions.
- Section 4.4: Classify possible Delone simplices, Delone polytopes, extreme Delone polytopes, in particular in dimension 7.
- Section 5.3: Analyze (and possibly improve) the currently best known covering lattices. Try to obtain series of lattices which contain new best known lattices in higher dimensions.
- Section 5.3: Use the techniques we used for covering lattices to find new best known (conjecturally best) lattice packing-coverings and quantizers (up to dimension 24).

$\ast$ Section 5.5: Prove that the Leech lattice gives a thinnest (lattice) sphere covering in 24-dimensions.

- Section 5.6: Prove that there are only two lattice covering maxima in dimension 7.
- Section 5.7: Study $T$-extreme Delone polytopes, in particular for linear subspaces $T$ containing PQFs of well-rounded lattices.

$\ast$ Section 5.7: Prove the covering conjecture (5.21) in all dimensions or find a counterexample.

$\ast$ Section 5.7: Prove the Minkowski conjecture 5.49 in all dimensions or find a counterexample.

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[286] scc *(secondary cone cruiser) by A. Schürmann and F. Vallentin*, ver. 1.0., see http://www.math.uni-magdeburg.de/lattice_geometry/.

[287] shvec *by F. Vallentin*, ver. 1.0., see http://www.math.uni-magdeburg.de/lattice_geometry/.

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# Notations

In the following we list some of the notations used at different places of this book and which may not be commonly known (among mathematicians). The listing is in a mixed alphabetical (by initial sound) and thematic order. The second column lists the page of the first occurrence.

| | | |
|---|---|---|
| $\langle \cdot, \cdot \rangle$ | 2, 37 | inner product on $\mathcal{S}^d$ or $\mathcal{S}^{d,m}$ |
| $\| \cdot \|$ | 5 | Euclidean norm |
| Aut | 33, 46, 92 | autom. group of PQF, periodic form or Delone subdiv. |
| $\mathrm{Aut}_{\boldsymbol{t}}$ | 49 | $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$-restricted automorphism group of PQF |
| $B^d$ | 5 | Euclidean unit ball in $\mathbb{R}^d$ |
| $\mathrm{BR}_{L,\beta}(Q)$ | 86 | matrix with property $\lvert \mathrm{BR}_{L,\beta}(Q)\rvert \geq 0 \Leftrightarrow \mu_L(Q) \leq \beta$ |
| $\mathcal{B}_{L,\leq\beta}$, $\mathcal{B}_{L,=\beta}$ | 92 | sets of PQFs with $\mu_L(Q) \leq \beta$ and $\mu_L(Q) = \beta$ |
| $\mathrm{cone}(M)$ | 29 | conic hull of a set $M$ |
| $\mathbf{D}_\alpha$ | 92 | set of PQFs $Q$ with $\det Q \geq \alpha$ |
| $\delta$ | 7, 9, 38 | packing density of discrete set, PQF or periodic form |
| $\delta_d$ | 8 | density of the densest lattice sphere packing in $\mathbb{R}^d$ |
| $\delta_{d,m}$ | 38 | density of the densest $m$-periodic sphere packing in $\mathbb{R}^d$ |
| $\delta_d^*$ | 9 | density of the densest sphere packing in $\mathbb{R}^d$ |
| det | 6, 38 | determinant of lattice, PQF (matrix) or periodic form |
| $\boldsymbol{\Delta}(\mathcal{D})$ | 57, 60 | secondary cone of Delone subdivision $\mathcal{D}$ |
| $\boldsymbol{\Delta}_T(\mathcal{D})$ | 67 | $T$-secondary cone $\boldsymbol{\Delta}(\mathcal{D}) \cap T$ with $T \subseteq \mathcal{S}^d$ linear |
| $\boldsymbol{\Delta}(\Lambda, P)$ | 74 | secondary cone of a polyhedron $P$ w.r.t. vertex-set $\Lambda$ |
| $\mathrm{dens}\,\Lambda$ | 10 | point density of a discrete set $\Lambda$ |
| $\mathrm{Del}_\Lambda(Q)$ | 57 | Delone subdivision of PQF $Q$ w.r.t. vertex-set $\Lambda$ |
| $\Gamma_d(N)$ | 46 | principal congruence subgroup of level $N$ |
| $\gamma$ | 13, 84 | packing-covering constant of discrete set or PQF |
| $\gamma_d$ | 13 | smallest lattice packing-covering constant in $\mathbb{R}^d$ |
| $\gamma_d^*$ | 13 | smallest packing-covering constant in $\mathbb{R}^d$ |
| $\mathsf{GL}_d(\mathbb{Z})$ | 2 | $\{U \in \mathbb{Z}^{d\times d} : \lvert \det U\rvert = 1\}$ |

| | | |
|---|---|---|
| $\mathsf{GL}_d(\mathbb{R})$ | 4 | $\{A \in \mathbb{R}^{d\times d} : \det A \neq 0\}$ |
| $\mathsf{GL}_d^{\boldsymbol{t}}(\mathbb{Z})$ | 46 | automorphism group of standard periodic set $\Lambda_{\boldsymbol{t}}$ |
| $g_{L,\beta}$ | 94 | gradient of smooth surface $\mathcal{B}_{L,=\beta}$ |
| $\mathcal{H}_d$ | 6 | Hermite's constant |
| $\lambda(L), \lambda(\Lambda)$ | 6, 9 | packing radius of lattice or discrete set |
| $\lambda(Q), \lambda(X)$ | 3, 38 | (generalized) arith. minimum of a PQF or periodic form |
| $\lambda_i$ | 21 | $i$-th successive minimum of lattice or PQF |
| $\Lambda_{\boldsymbol{t}}$ | 65 | standard periodic vertex-set defined by $\boldsymbol{t}$ |
| $\mathcal{M}_d, \mathcal{M}_d^+$ | 19 | Minkowski's reduction domains |
| $\operatorname{Min} Q, \operatorname{Min} X$ | 3 , 38 | representatives of (general.) arith. minimum $\lambda(Q), \lambda(X)$ |
| $\mu(\Lambda)$ | 11 | covering radius of discrete set $\Lambda$ |
| $\mu(Q), \mu_\Lambda(Q)$ | 83, 84 | inhom. minimum of PQF w.r.t. $\mathbb{Z}^d$, respectively $\Lambda$ |
| $\mu_P(Q)$ | 86, 87 | squared circumradius of $P$ with respect to $Q$ |
| $N_{V,\boldsymbol{w}}$ | 57 | quadratic form defining regulator |
| $N_{L,L'}$ | 57 | quadratic form def. regulator for adj. Del. simplices $L, L'$ |
| $\mathcal{N}_{\Lambda,P}(Q)$ | 125 | normal cone of Baranovskii cone $\boldsymbol{\Delta}(\Lambda, P)$ at $Q$ |
| $\mathsf{O}_d(\mathbb{R})$ | 5 | $\{O \in \mathbb{R}^{d\times d} : O^tO = \mathrm{id}_d\}$ |
| $\mathcal{P}(Q), \mathcal{P}(X)$ | 31, 40 | dual cones of (generalized) Voronoi domain $\mathcal{V}(Q), \mathcal{V}(X)$ |
| $\mathcal{P}_\lambda$ | 27 | Ryshkov polyhedron |
| $\mathcal{P}_{m,\lambda}$ | 38 | generalized Ryshkov set |
| $\mathcal{P}_{\lambda,\boldsymbol{t}}$ | 47 | Ryshkov polyhedron of standard periodic set $\Lambda_{\boldsymbol{t}}$ |
| $p_{i,j,\boldsymbol{v}}(X)$ | 38 | polynomial $Q[\boldsymbol{t}_i - \boldsymbol{t}_j - \boldsymbol{v}]$ for periodic form $X = (Q, \boldsymbol{t})$ |
| $\mathcal{S}^d$ | 1 | $\{Q \in \mathbb{R}^{d\times d} : Q^t = Q\}$ |
| $\mathcal{S}^d_{>0}$ | 2 | $\{Q \in \mathcal{S}^d : Q$ positive definite$\}$ |
| $\mathcal{S}^d_{\geq 0}$ | 4 | $\{Q \in \mathcal{S}^d : Q$ positive semi-definite$\}$ |
| $\tilde{\mathcal{S}}^d_{\geq 0}$ | 35 | rational closure of $\mathcal{S}^d_{>0}$ |
| $\mathcal{S}^{d,m}_{>0}$ | 37 | parameter space for $m$-periodic point sets in $\mathbb{R}^d$ |
| $\mathcal{S}^{d,m}$ | 37 | Euclidean space $\mathcal{S}^d \times \mathbb{R}^{d\times(m-1)}$ containing $\mathcal{S}^{d,m}_{>0}$ |
| $\Theta$ | 11, 83 | covering density of discrete set or PQF |
| $\Theta_d$ | 11 | density of thinnest lattice covering in $\mathbb{R}^d$ |
| $\Theta_d^*$ | 11 | density of thinnest covering in $\mathbb{R}^d$ |
| $\Theta_P$ | 123 | covering density function of Delone polytope $P$ |
| $T_G$ | 34 | linear space of $G$-invariant quadratic forms |
| $\mathcal{V}(Q), \mathcal{V}(X)$ | 29, 40 | (generalized) Voronoi domain |
| $\mathcal{W}_{\mathcal{D},\leq\beta}$ | 92 | set of PQFs in $\boldsymbol{\Delta}(\mathcal{D})$ with $\mu(Q) \leq \beta$ |

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Starting from classical arithmetical questions on quadratic forms, this book takes the reader step by step through the connections with lattice sphere packing and covering problems. As a model for polyhedral reduction theories of positive definite quadratic forms, Minkowski's classical theory is presented, including an application to multidimensional continued fraction expansions. The reduction theories of Voronoi are described in great detail, including full proofs, new views, and generalizations that cannot be found elsewhere. Based on Voronoi's second reduction theory, the local analysis of sphere coverings and several of its applications are presented. These include the classification of totally real thin number fields, connections to the Minkowski conjecture, and the discovery of new, sometimes surprising, properties of exceptional structures such as the Leech lattice or the root lattices.

Throughout this book, special attention is paid to algorithms and computability, allowing computer-assisted treatments. Although dealing with relatively classical topics that have been worked on extensively by numerous authors, this book is exemplary in showing how computers may help to gain new insights.

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